

INDECOMPOSABLE REPRESENTATIONS OF QUIVERS ON INFINITE-DIMENSIONAL HILBERT SPACES

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ABSTRACT. We study indecomposable representations of quivers on separable infinite-dimensional Hilbert spaces by bounded operators. We consider a complement of Gabriel's theorem for these representations. Let Γ be a finite, connected quiver. If its underlying undirected graph contains one of extended Dynkin diagrams \tilde{A}_n ($n \geq 0$), \tilde{D}_n ($n \geq 4$), \tilde{E}_6, \tilde{E}_7 and \tilde{E}_8 , then there exists an indecomposable representation of Γ on separable infinite-dimensional Hilbert spaces.

KEYWORDS: quiver, indecomposable representation, Dynkin diagram, reflection functor, Hilbert space.

AMS SUBJECT CLASSIFICATION: 46C07, 47A15, 15A21, 16G20, 16G60.

1. INTRODUCTION

We studied the relative position of *several subspaces* in a separable infinite-dimensional Hilbert space in [EW]. In this paper we extend it to the relative position of several subspaces along quivers. More generally we study representations of quivers on infinite-dimensional Hilbert spaces by bounded operators. We call them Hilbert representations for short.

Gabriel's theorem says that a connected finite quiver has only finitely many indecomposable representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8 [Ga]. The theory of representations of quivers on finite-dimensional vector spaces has been developed by Bernstein-Gelfand-Ponomarev [BGP], Donovan-Freislich [DF], V. Dlab-Ringel [DR], Gabriel-Roiter [GR], Kac [Ka], Nazarova [Na]

Furthermore locally scalar representations of quivers in the category of Hilbert spaces were introduced by Kruglyak and Roiter [KR]. They associate operators and their adjoint operators with arrows and classify them up to the unitary equivalence. They proved an analog of Gabriel's theorem. Their study is connected with representations of *-algebras generated by linearly related orthogonal projections, see for example, S. Kruglyak, V. Rabanovich and Y. Samoilenko [KRS].

In this paper we study the existence of indecomposable representations of quivers on infinite-dimensional Hilbert spaces. We associate bounded operators with arrows but we do not associate their adjoint operators simultaneously as in [KR0].

In particular if we consider a certain quiver Γ whose underlying undirected graph is the extended Dynkin diagram \tilde{D}_4 , then indecomposability of Hilbert representations of Γ is reduced to indecomposability of systems of four subspaces studied in [EW]. We consider a complement of Gabriel's theorem for Hilbert representations and prove one direction: If the underlying undirected graph of a finite, connected quiver Γ contains one of extended Dynkin diagrams \tilde{A}_n ($n \geq 0$), \tilde{D}_n ($n \geq 4$), \tilde{E}_6, \tilde{E}_7 and \tilde{E}_8 , then there exists an indecomposable representation of Γ on separable infinite-dimensional Hilbert spaces. The result does not depend on the choice of orientation. But we cannot prove the converse. In fact if the converse were true, then a long standing problem in [Ha] on transitive lattices of subspaces of Hilbert spaces would be settled.

Recall that we study relative position of n subspaces in a separable infinite-dimensional Hilbert space in [EW]. See Y. P. Moskaleva and Y. S. Samoilenko [MS] on a connection with $*$ -algebras generated by projections. Let H be a Hilbert space and E_1, \dots, E_n be n subspaces in H . Then we say that $\mathcal{S} = (H; E_1, \dots, E_n)$ is a system of n subspaces in H or a n -subspace system in H . A system \mathcal{S} is called indecomposable if \mathcal{S} can not be decomposed into a nontrivial direct sum. For any bounded linear operator A on a Hilbert space K , we can associate a system \mathcal{S}_A of four subspaces in $H = K \oplus K$ by

$$\mathcal{S}_A = (H; K \oplus 0, 0 \oplus K, \text{graph } A, \{(x, x); x \in K\}).$$

In particular on a finite dimensional space, Jordan blocks correspond to indecomposable systems. Moreover on an infinite dimensional Hilbert space, the above system \mathcal{S}_A is indecomposable if and only if A is strongly irreducible, which is an infinite-dimensional analog of a Jordan block, see books by Jiang and Wang [JW], [JW2]. For example, a unilateral shift operator is a typical example of strongly irreducible operator. Such a system of four subspaces give an indecomposable Hilbert representation of a quiver with underlying undirected graph \tilde{D}_4 . We transform these representations and make up indecomposable Hilbert representations of other quivers in this paper. In finite dimensional case many such functors are introduced, see [DF], for example. We follow some of their constructions. But we have not yet proved all such functors preserve indecomposability in infinite-dimensional Hilbert setting in general. We have checked the indecomposability of the Hilbert representations in our concrete examples by our method.

Main theorem of the paper is the following: Let Γ be a finite, connected quiver. If its underlying undirected graph contains one of extended Dynkin diagrams \tilde{A}_n ($n \geq 0$), \tilde{D}_n ($n \geq 4$), \tilde{E}_6, \tilde{E}_7 and \tilde{E}_8 ,

then there exists an indecomposable representation of Γ on separable infinite-dimensional Hilbert spaces. There were two difficulties which did not appear in finite-dimensional case. Firstly we need to find indecomposable, infinite-dimensional representations of a certain class of Γ . We constructed them by studying the relative position of several subspaces along quivers, where vertices and arrows are represented by subspaces and natural inclusion maps. Secondly we need to change the orientation of the quiver preserving indecomposability. Here comes reflection functors. Being different from finite-dimensional case, we need to check the co-closedness condition at sources to show that indecomposability is preserved under reflection functors. We introduce a certain nice class, called positive-unitary diagonal Hilbert representations, such that co-closedness is easily checked and preserved under reflection functors at any source.

We believe that there exists an analogy between study of Hilbert representations of quivers and subfactor theory invented by V. Jones [J]. In fact Dynkin diagrams also appear in the classification of subfactors, see, for example, Goodman, de la Harpe and Jones [GHJ], Evans and Kawahigashi [EK]. But we have not yet understood the full relations between them.

There exists a close interplay between finite-dimensional representations of quivers and finite-dimensional representations of path algebras in purely algebraic sense. Any Hilbert representation of a quiver gives an operator algebra representation of the corresponding path algebra. Therefore we expect some relation between Hilbert representations of quivers and certain operator algebras associated with quivers. There exist some related works, see P. Muhly [Mu], D. W. Kribs and S. C. Power [KP] and B. Solel [S]. But the relation is not so clear for us.

Throughout the paper a projection means an operator e with $e^2 = e = e^*$ and an idempotent means an operator p with $p^2 = p$.

In purely algebraic setting, it is known that if a finite-dimensional algebra R is not of representation-finite type, then there exist indecomposable R -modules of infinite length as in M. Auslander [Au]. Since we consider bounded operator representations on Hilbert spaces, the result in [Au] cannot be applied directly. See a book [KR] for infinite length modules.

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2. REPRESENTATIONS OF QUIVERS

A quiver $\Gamma = (V, E, s, r)$ is a quadruple consisting of the set V of vertices, the set E of arrows, and two maps $s, r : E \rightarrow V$, which associate with each arrow $\alpha \in E$ its support $s(\alpha)$ and range $r(\alpha)$. We sometimes denote by $\alpha : x \rightarrow y$ an arrow with $x = s(\alpha)$ and $y = r(\alpha)$. Thus a quiver is just a directed graph. We denote by $|\Gamma|$ the underlying

undirected graph of a quiver Γ . A quiver Γ is said to be connected if $|\Gamma|$ is a connected graph. A quiver Γ is said to be finite if both V and E are finite sets.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that (H, f) is a *Hilbert representation* of Γ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_\alpha)_{\alpha \in E}$ is a family of bounded linear operators $f_\alpha : H_{s(\alpha)} \rightarrow H_{r(\alpha)}$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (H, f) and (K, g) be Hilbert representations of Γ . A *homomorphism* $T : (H, f) \rightarrow (K, g)$ is a family $T = (T_v)_{v \in V}$ of bounded operators $T_v : H_v \rightarrow K_v$ satisfying, for any arrow $\alpha \in E$

$$T_{r(\alpha)} f_\alpha = g_\alpha T_{s(\alpha)}.$$

The composition $T \circ S$ of homomorphisms T and S is defined by $(T \circ S)_v = T_v \circ S_v$ for $v \in V$. Thus we have obtained a category $HRep(\Gamma)$ of Hilbert representations of Γ

We denote by $Hom((H, f), (K, g))$ the set of homomorphisms $T : (H, f) \rightarrow (K, g)$. We denote by $End(H, f) := Hom((H, f), (H, f))$ the set of endomorphisms. We denote by

$$Idem(H, f) := \{T \in End(H, f) \mid T^2 = T\}$$

the set of idempotents of $End(H, f)$. Let $0 = (0_v)_{v \in V}$ be a family of zero endomorphisms 0_v and $I = (I_v)_{v \in V}$ be a family of identity endomorphisms I_v . The both 0 and I are in $Idem(H, f)$.

Let $\Gamma = (V, E, s, r)$ be a finite quiver and $(H, f), (W, g)$ be Hilbert representations of Γ . We say that (H, f) and (W, g) are *isomorphic*, denoted by $(H, f) \simeq (W, g)$, if there exists an isomorphism $\varphi : (H, f) \rightarrow (W, g)$, that is, there exists a family $\varphi = (\varphi_v)_{v \in V}$ of bounded invertible operators $\varphi_v \in B(H_v, K_v)$ such that, for any arrow $\alpha \in E$,

$$\varphi_{r(\alpha)} f_\alpha = g_\alpha \varphi_{s(\alpha)}.$$

We say that (H, f) is a finite-dimensional representation if H_v is finite-dimensional for all $v \in V$. And (H, f) is an infinite-dimensional representation if H_v is infinite-dimensional for some $v \in V$.

3. INDECOMPOSABLE REPRESENTATIONS OF QUIVERS

In this section we shall introduce a notion of indecomposable representation, that is, a representation which cannot be decomposed into a direct sum of smaller representations anymore.

Definition.(Direct sum) Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (K, g) and (K', g') be Hilbert representations of Γ . Define the direct sum $(H, f) = (K, g) \oplus (K', g')$ by

$$H_v = K_v \oplus K'_v \text{ (for } v \in V \text{) and } f_\alpha = g_\alpha \oplus g'_\alpha \text{ (for } \alpha \in E \text{).}$$

We say that a Hilbert representation (H, f) is zero, denoted by $(H, f) = 0$, if $H_v = 0$ for any $v \in E$.

Definition.(Indecomposable representation). A Hilbert representation (H, f) of Γ is called *decomposable* if (H, f) is isomorphic to a direct sum of two non-zero Hilbert representations. A non-zero Hilbert representation (H, f) of Γ is said to be *indecomposable* if it is not decomposable, that is, if $(H, f) \cong (K, g) \oplus (K', g')$ then $(K, g) \cong 0$ or $(K', g') \cong 0$.

We start with an easy fact. Let H be a Hilbert space and K_1, K_2 be closed subspaces of H . Assume that $K_1 \cap K_2 = 0$ and $H = K_1 + K_2$. But we do not assume that K_1 and K_2 are orthogonal. Let $T : H \rightarrow H$ be a bounded operator with $TK_i \subset K_i$ for $i = 1, 2$. Define $S_i = T|_{K_i} : K_i \rightarrow K_i$. Consider the (orthogonal) direct sum $K_1 \oplus K_2$ and the bounded operator $S_1 \oplus S_2$ on $K_1 \oplus K_2$. Define a bounded invertible operator $\varphi : H \rightarrow K_1 \oplus K_2$ by $\varphi(h) = (h_1, h_2)$ for $h = h_1 + h_2$ with $h_i \in K_i$, as in the proof of [EW, Lemma 2.1.] Then we have $T = \varphi^{-1} \circ (S_1 \oplus S_2) \circ \varphi$.

The following proposition is used frequently to show the indecomposability in concrete examples.

Proposition 3.1. *Let (H, f) be a Hilbert representation of a quiver Γ . Then the following conditions are equivalent:*

- (1) (H, f) is indecomposable.
- (2) $\text{Idem}(H, f) = \{0, I\}$.

Proof. $\neg(1) \implies \neg(2)$: Assume that (H, f) is not indecomposable. Then there exist non-zero representations (K, g) and (K', g') of Γ , such that $(H, f) \cong (K, g) \oplus (K', g')$. For any $x \in V$, define the projection $Q_x \in B(K_x \oplus K'_x)$ of $K_x \oplus K'_x$ onto K_x . Then $Q := (Q_x)_{x \in V}$ is in $\text{End}(K \oplus K', g \oplus g')$, because

$$Q_{r(\alpha)}(g_\alpha, g'_\alpha) = (g_\alpha, 0) = (g_\alpha, g'_\alpha)Q_{s(\alpha)}$$

for any $\alpha \in E$. Since there exists $v, w \in E$ such that $K_v \neq 0$ and $K'_w \neq 0$, we have $Q_v \neq 0$ and $Q_w \neq I$. Thus $Q \neq 0$ and $Q \neq I$. Let $\varphi = (\varphi_x)_{x \in V} : (H, f) \rightarrow (K, g) \oplus (K', g')$ be an isomorphism. Put $P_x = (\varphi_x)^{-1}Q_x\varphi_x$ for $x \in V$ and $P := (P_x)_{x \in V} \in \text{Idem}(H, f)$. Then $P \neq 0$ and $P \neq I$.

$\neg(2) \implies \neg(1)$: Assume that there exists $P \in \text{Idem}(H, f)$ with $P \neq 0$ and $P \neq I$. Thus there exist $v \in V$ and $w \in V$ such that $P_v \neq 0_v$, $P_w \neq I_w$. For any $x \in V$, define closed subspaces

$$K_x = P_x(H_x), \text{ and } K'_x = (I - P_x)(H_x).$$

Then $K := (K_x)_x \neq 0$, $K' := (K'_x)_x \neq 0$ and $H \cong K \oplus K'$. For any $\alpha \in E$, let $x = s(\alpha)$ and $y = r(\alpha)$. Since $f_\alpha P_x = P_y f_\alpha$, we have $f_\alpha K_x \subset K_y$. Similarly, $f_\alpha (I - P_x) = (I - P_y) f_\alpha$ implies that $f_\alpha K'_x \subset K'_y$. We can define $g_\alpha = f_\alpha|_{K_x} : K_x \rightarrow K_y$ and $g'_\alpha = f_\alpha|_{K'_x} : K'_x \rightarrow K'_y$. Put $g = (g_\alpha)_\alpha$ and $g' = (g'_\alpha)_\alpha$. Then (K, g) and (K', g') are representations of Γ . Define $\varphi_x : H_x \rightarrow K_x \oplus K'_x$ by $\varphi_x(\xi) = (P_x \xi, (I - P_x)\xi)$ for $\xi \in H_x$.

Then $\varphi := (\varphi_x)_{x \in V} : (H, f) \rightarrow (K, g) \oplus (K', g')$ is an isomorphism. Since $K := (K_x)_{x \in V} \neq 0$ and $K' := (K'_x)_{x \in V} \neq 0$, (H, f) is decomposable. \square

Remark.(1) The proof of the above Proposition 3.1 shows that (H, f) is decomposable if and only if there exist non-zero families $K = (K_x)_{x \in V}$ and $K' = (K'_x)_{x \in V}$ of closed subspaces K_x and K'_x of H_x with $K_x \cap K'_x = 0$ and $K_x + K'_x = H_x$ such that $f_\alpha K_x \subset K_y$ and $f_\alpha K'_x \subset K'_y$ for any arrow $\alpha : x \rightarrow y$.

(2) In the statement of the above Proposition 3.1, we cannot replace the set $Idem(H, f)$ of idempotents of endomorphisms by the set of projections of endomorphisms. For example, let $H_0 = \mathbb{C}^2$. Fix an angle θ with $0 < \theta < \pi/2$. Put $H_1 = \mathbb{C}(1, 0)$ and $H_2 = \mathbb{C}(\cos\theta, \sin\theta)$. Then the system $(H_0; H_1, H_2)$ of two subspaces is isomorphic to

$$(\mathbb{C}^2; \mathbb{C} \oplus 0, 0 \oplus \mathbb{C}) \cong (\mathbb{C}; \mathbb{C}, 0) \oplus (\mathbb{C}; 0, \mathbb{C}).$$

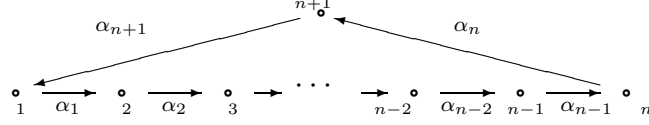
Hence $(H_0; H_1, H_2)$ is decomposable. See Example 2 in [EW] and the Remark after it. Now consider the following quiver Γ :

$$\circ_1 \xrightarrow{\alpha_1} \circ_0 \xleftarrow{\alpha_2} \circ_2$$

Define a Hilbert representation (H, f) of Γ by $H = (H_i)_{i=0,1,2}$ and canonical inclusion maps $f_i = f_{\alpha_i} : H_i \rightarrow H_0$ for $i = 1, 2$. Then the Hilbert representation (H, f) is also decomposable, see Example 3 below in this paper. But for any $P = (P_i)_{i=0,1,2} \in End(H, f)$, if $P_i \in B(H_i)$ is a projection for $i = 0, 1, 2$, then $P = 0$ or $P = I$. In fact $P_0(H_i) \subset H_i$ for $i = 1, 2$. Let $e_1 \in B(H_0)$ and $e_2 \in B(H_0)$ be the projections of H_0 onto H_1 and H_2 . Then the C^* -algebra $C^*(\{e_1, e_2\})$ generated by e_1 and e_2 is exactly $B(H_0) \cong M_2(\mathbb{C})$. Since P_0 commutes with e_1 and e_2 , $P_0 = 0$ or $P_0 = I$. Because $P_i = P_0|_{H_i}$, $P_i = 0$ or $P_i = I$ simultaneously.

Example 1. Let Γ be a loop with one vertex 1 and one arrow $\alpha : 1 \rightarrow 1$, that is, the underlying undirected graph is an extended Dynkin diagram \tilde{A}_0 . Let $H_1 = \ell^2(\mathbb{N})$ and $f_\alpha = S : H_1 \rightarrow H_1$ be a unilateral shift. Then the Hilbert representation (H, f) is infinite-dimensional and indecomposable. In fact, any $T \in Idem(H, f)$ can be identified with $T \in B(\ell^2(\mathbb{N}))$ with $T^2 = T$ and $TS = ST$. Since T commutes with a unilateral shift S , the operator T is a lower triangular Toeplitz matrix. Since T is an idempotent, $T = 0$ or $T = I$. Thus (H, f) is indecomposable. Replacing S by $S + \lambda I$ for $\lambda \in \mathbb{C}$, we obtain a family of infinite-dimensional, indecomposable Hilbert representations (H^λ, f^λ) of Γ . Since (H^λ, f^λ) and (H^μ, f^μ) are isomorphic if and only if $S + \lambda I$ and $S + \mu I$ is similar, we have uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ .

Example 2. Let $\Gamma = (V, E, s, r)$ be a quiver whose underlying undirected graph is an extended Dynkin diagram \tilde{A}_n , ($n \geq 1$). Then there exist uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ . For example, consider

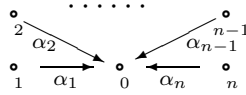


Define a Hilbert representation (H, f) of Γ by $H_1 = H_2 = \dots = H_{n+1} = \ell^2(\mathbb{N})$, $f_{\alpha_2} = f_{\alpha_3} = \dots = f_{\alpha_{n+1}} = I$ and $f_{\alpha_1} = S$, the unilateral shift. Let $P = (P_k)_{k \in V} \in \text{Idem}(H, f)$. Then

$$P_1 = P_2 = \dots = P_{n+1} \quad \text{and} \quad SP_1 = P_2S.$$

Since P_1 is an idempotent and $SP_1 = P_1S$, we have $P_1 = 0$ or $P_1 = I$. This implies $P = 0$ or $P = I$. Therefore (H, f) is indecomposable. Replacing S by $S + \lambda I$ for $\lambda \in \mathbb{C}$, we obtain uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ .

Example 3. Let L be a Hilbert space and E_1, \dots, E_n be n subspaces in L . Then we say that $\mathcal{S} = (L; E_1, \dots, E_n)$ is a system of n subspaces in L . A system \mathcal{S} is called indecomposable if \mathcal{S} cannot be decomposed into a non-trivial direct sum, see [EW]. Consider the following quiver $\Gamma_n = (V, E, s, r)$



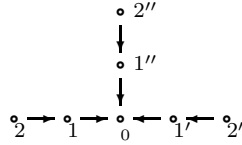
Define a Hilbert representation (H, f) of Γ_n by $H_k := E_k$ ($k = 1, \dots, n$), $H_0 := L$ and $f_k = f_{\alpha_k} : H_k = E_k \rightarrow H_0 = L$ be the inclusion map. Then the system \mathcal{S} of n subspaces is indecomposable if and only if the corresponding Hilbert representation (H, f) of Γ is indecomposable. In fact, assume that \mathcal{S} is indecomposable. Let $P = (P_k)_{k \in V} \in \text{Idem}(H, f)$. Then $f_k P_k = P_0 f_k$. This implies $P_0(H_k) \subset H_k$ for $k = 1, \dots, n$. Since P_0 is idempotent and \mathcal{S} is indecomposable, $P_0 = 0$ or $P_0 = I$ by [EW, Lemma 3.2]. Since $f_k P_k = P_0 f_k$, $P_k = 0$ or $P_k = I$ simultaneously. Thus $P = 0$ or $P = I$, that is, (H, f) is indecomposable. Conversely assume that (H, f) is indecomposable. Let $R \in B(L)$ be an idempotent with $R(E_k) \subset E_k$ for $k = 1, \dots, n$. Define $P = (P_k)_{k \in V}$ by $P_0 = R$ and $P_k = P_0|_{H_k}$. Then $P \in \text{Idem}(H, f)$. Therefore $P = 0$ or $P = I$. Thus $R = 0$ or $R = I$. Hence \mathcal{S} is indecomposable.

We can also show that two systems \mathcal{S} and \mathcal{S}' of n subspaces are isomorphic if and only if the corresponding Hilbert representations (H, f) and (H', f') of Γ are isomorphic.

Since there exist uncountably many, indecomposable systems of four subspaces in an infinite-dimensional Hilbert space as in [EW], there exist uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ_4 whose underlying undirected graph is the extended Dynkin diagram \tilde{D}_4 .

In particular, let $K = \ell^2(\mathbb{N})$ and $A \in B(K)$ be a strongly irreducible operator studied in [JW], [JW2] for example, a unilateral shift. Define $H_0 = K \oplus K$, $H_1 = K \oplus 0$, $H_2 = 0 \oplus K$, $H_3 = \{(x, Ax) \in K \oplus K | x \in K\}$, $H_4 = \{(x, x) \in K \oplus K | x \in K\}$. Let $f_k = f_{\alpha_k} : H_k \rightarrow H_0$ be the inclusion map for $k = 1, 2, 3, 4$. Put $H^{(A)} = (H_v)_{v \in V}$ and $f^{(A)} = (f_\alpha)_{\alpha \in E}$. Then $(H^{(A)}, f^{(A)})$ is an infinite-dimensional, indecomposable Hilbert representation of \tilde{D}_4 . Moreover let A and B be strongly irreducible operators on $\ell^2(\mathbb{N})$. Then two indecomposable Hilbert representations $(H^{(A)}, f^{(A)})$ and $(H^{(B)}, f^{(B)})$ of \tilde{D}_4 are isomorphic if and only if two operators A and B are similar.

Example 4. Consider the following quiver $\Gamma = (V, E, s, r)$



Then underlying undirected graph is an extended Dynkin diagram \tilde{E}_6 . Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K . We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:

Put $H_0 = K \oplus K \oplus K$, $H_1 = K \oplus 0 \oplus K$, $H_2 = 0 \oplus 0 \oplus K$, $H_{1'} = K \oplus K \oplus 0$, $H_{2'} = 0 \oplus K \oplus 0$, $H_{1''} = \{(x, x, x) + (y, Sy, 0) \in K^3 | x, y \in K\}$ and $H_{2''} = \{(x, x, x) \in K^3 | x \in K\}$.

Then $H_{1''}$ is a closed subspace of H_0 . In fact, let

$$(x_n, x_n, x_n) + (y_n, Sy_n, 0) = (x_n + y_n, x_n + Sy_n, x_n) \in H_{1''}$$

converges to $(a, b, c) \in H_0$. Then $x_n \rightarrow c$, $y_n \rightarrow a - c$ and $c + S(a - c) = b$. Define $x = c$ and $y = a - c$. Then $(a, b, c) = (x, x, x) + (y, Sy, 0) \in H_{1''}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \rightarrow H_{r(\alpha)}$ be the canonical inclusion map. We shall show that the Hilbert representation (H, f) is indecomposable. Take $T = (T_v)_{v \in V} \in \text{Idem}(H, f)$. Since $T \in \text{End}(H, f)$, for any $v \in \{1, 2, 1', 2', 1'', 2''\}$ and any $x \in H_v$, we have $T_0 x = T_v x$. In particular, $T_0 H_v \subset H_v$. Since $H_1 \cap H_{1'} = K \oplus 0 \oplus 0$, $H_{2'} = 0 \oplus K \oplus 0$ and $H_2 = 0 \oplus 0 \oplus K$, T_0 preserves these subspaces. Hence T_0 is a block diagonal operator with $T_0 = P \oplus Q \oplus R \in B(K \oplus K \oplus K)$.

Since $T_0(H_{2''}) \subset H_{2''}$, for any $x \in K$,

$$T_0(x, x, x) = (y, y, y)$$

for some $y \in K$. Therefore $P = Q = R$ and $T_0 = P \oplus P \oplus P$. Moreover P is an idempotent, because so is T_0 . Since T_0 preserves

$H_{1'} \cap H_{1''} = \{(y, Sy, 0) \in K^3 \mid y \in K\}$, for any $y \in K$, there exists $z \in K$ such that

$$T_0 \begin{pmatrix} y \\ Sy \\ 0 \end{pmatrix} = \begin{pmatrix} Py \\ PSy \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ Sz \\ 0 \end{pmatrix}.$$

Therefore $PSy = Sz = SPy$ for any $y \in K$, i.e., $PS = SP$. Since P is an idempotent, $P = 0$ or $P = I$. This means that $T_0 = 0$ or $T_0 = I$. Because $T_0x = T_vx$ for any $x \in H_v$ for $v \in \{1, 2, 1', 2', 1'', 2''\}$, we have $T_v = 0$ or $T_v = I$ simultaneously. Thus $T = 0$ or $T = I$, that is, $\text{Idem}(H, f) = \{0, I\}$. Therefore (H, f) is indecomposable.

Example 5. We have a different kind of infinite-dimensional, indecomposable Hilbert representation $(L, g) = ((L_v)_{v \in V}, (g_\alpha)_{\alpha \in E})$ of the same Γ in Example 4 as follows: Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K . Define $L_0 = K \oplus K \oplus K$, $L_1 = 0 \oplus K \oplus K$, $L_2 = 0 \oplus \{(y, Sy) \in K^2 \mid y \in K\}$, $L_{1'} = K \oplus K \oplus 0$, $L_{2'} = \{(x, x) \in K^2 \mid x \in K\} \oplus 0$, $L_{1''} = K \oplus 0 \oplus K$, $L_{2''} = \{(x, 0, x) \in K^3 \mid x \in K\}$. For any arrow $\alpha \in E$, let $g_\alpha : L_{s(\alpha)} \rightarrow L_{r(\alpha)}$ be the canonical inclusion map. We can similarly prove that the Hilbert representation (L, g) is indecomposable.

We shall show that two Hilbert representations in Example 4 and 5 are not isomorphic. In fact, on the contrary, suppose that there were an isomorphism $\varphi = (\varphi_v)_{v \in V} : (H, f) \rightarrow (L, g)$. Since any arrow is represented by the canonical inclusion, $\varphi_0 : H_0 \rightarrow L_0$ satisfies that $\varphi_v = \varphi_0|_{H_v} : H_v \rightarrow L_v$. This implies that $\varphi_0(H_v) \subset L_v$ for any $v \in V$. Since $\varphi_0(H_{1'}) \subset L_{1'}$ and $\varphi_0(H_1) \subset L_1$, φ_0 has a form such that

$$\varphi_0 = \begin{pmatrix} 0 & A & 0 \\ B & C & D \\ 0 & 0 & E \end{pmatrix}.$$

Since $\varphi_0(H_2) \subset L_2$, for any $z \in K$ there exists $y \in K$ such that $(0, Dz, Ez) = (0, y, Sy)$. Hence $Ez = Sy = SDz$, so that $E = SD$. Then $\text{Im } \varphi_0 \subset K \oplus K \oplus \text{Im } S \neq L_0$. This contradicts the assumption that $\varphi_0 : H_0 \rightarrow L_0$ is onto. Therefore Hilbert representations (H, f) and (L, g) of Γ are not isomorphic.

4. REFLECTION FUNCTORS

Reflection functors are crucially used in the proof the classification of finite-dimensional, indecomposable representations of tame quivers. In fact any indecomposable representations of tame quivers can be reconstructed by iterating reflection functors on simple indecomposable representations. We can not expect such a best situation in infinite-dimensional Hilbert representations. But reflection functors are still useful to show that some property of representations of quivers on infinite-dimensional Hilbert spaces does not depend on the choice of

orientations and does depend on the fact underlying undirected graphs are (extended) Dynkin diagrams or not.

Let $\Gamma = (V, E, s, r)$ be a finite quiver. A vertex $v \in V$ is called a *sink* if $v \neq s(\alpha)$ for any $\alpha \in E$. Put $E^v = \{\alpha \in E \mid r(\alpha) = v\}$. We denote by \overline{E} the set of all formally reversed new arrows $\overline{\alpha}$ for $\alpha \in E$. Thus if $\alpha : x \rightarrow y$ is an arrow, then $\overline{\alpha} : x \leftarrow y$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we construct a new quiver $\sigma_v^+(\Gamma) = (\sigma_v^+(V), \sigma_v^+(E), s, r)$ as follows: All the arrows of Γ having v as range are reversed and all the other arrows remain unchanged. More precisely,

$$\sigma_v^+(V) = V \quad \sigma_v^+(E) = (E \setminus E^v) \cup \overline{E^v},$$

where $\overline{E^v} = \{\overline{\alpha} \mid \alpha \in E^v\}$.

Definition. (reflection functor Φ_v^+ .) Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we define a *reflection functor* at v

$$\Phi_v^+ : HRep(\Gamma) \rightarrow HRep(\sigma_v^+(\Gamma))$$

between the categories of Hilbert representations of Γ and $\sigma_v^+(\Gamma)$ as follows: For a Hilbert representation (H, f) of Γ , we shall define a Hilbert representation $(K, g) = \Phi_v^+(H, f)$ of $\sigma_v^+(\Gamma)$. Let

$$h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v$$

be a bounded linear operator defined by

$$h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha).$$

Define

$$K_v := \text{Ker } h_v = \{(x_\alpha)_{\alpha \in E^v} \in \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \mid \sum_{\alpha \in E^v} f_\alpha(x_\alpha) = 0\}.$$

Consider also the canonical inclusion map $i_v : K_v \rightarrow \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

For $\beta \in E^v$, let

$$P_\beta : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_{s(\beta)}$$

be the canonical projection. Then define

$$g_{\overline{\beta}} : K_{s(\overline{\beta})} = K_v \rightarrow K_{r(\overline{\beta})} = H_{s(\beta)} \quad \text{by} \quad g_{\overline{\beta}} = P_\beta \circ i_v$$

that is, $g_{\overline{\beta}}((x_\alpha)_{\alpha \in E^v}) = x_\beta$.

For $\beta \notin E^v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H, f) \rightarrow (H', f')$, we shall define a homomorphism

$$S = (S_u)_{u \in V} = \Phi_v^+(T) : (K, g) = \Phi_v^+(H, f) \rightarrow (K', g') = \Phi_v^+(H', f')$$

If $u = v$, a bounded operator $S_v : K_v \rightarrow K'_v$ is given by

$$S_v((x_\alpha)_{\alpha \in E^v}) = (T_{s(\alpha)}(x_\alpha))_{\alpha \in E^v}.$$

It is easy to see that S_v is well-defined and we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_v & \xrightarrow{i_v} & \bigoplus_{\alpha \in E^v} H_{s(\alpha)} & \xrightarrow{h_v} & H_v \\
& & S_v \downarrow & & (T_{s(\alpha)})_{\alpha \in E^v} \downarrow & & T_v \downarrow \\
0 & \longrightarrow & K'_v & \xrightarrow{i'_v} & \bigoplus_{\alpha \in E^v} H'_{s(\alpha)} & \xrightarrow{h'_v} & H'_v
\end{array}$$

For other $u \in V$ with $u \neq v$, we put

$$S_u = T_u : K_u = H_u \rightarrow K'_u = H'_u.$$

We shall consider a dual of the above construction. A vertex $v \in V$ is called a *source* if $v \neq r(\alpha)$ for any $\alpha \in E$. Put $E_v = \{\alpha \in E \mid s(\alpha) = v\}$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we construct a new quiver $\sigma_v^-(\Gamma) = (\sigma_v^-(V), \sigma_v^-(E), s, r)$ as follows: All the arrows of Γ having v as source are reversed and all the other arrows remain unchanged. More precisely,

$$\sigma_v^-(V) = V \quad \sigma_v^-(E) = (E \setminus E_v) \cup \overline{E_v},$$

where $\overline{E_v} = \{\overline{\alpha} \mid \alpha \in E_v\}$.

In order to define a reflection functor at a source, it is convenient to consider the orthogonal complement M^\perp of a closed subspace M of a Hilbert space H instead of the quotient H/M . Define an isomorphism $f : M^\perp \rightarrow H/M$ by $f(y) = [y] = y + M$ for $y \in M^\perp \subset H$. Then the inverse $f^{-1} : H/M \rightarrow M^\perp$ is given by $f^{-1}([x]) = P_M^\perp(x)$ for $x \in H$, where P_M^\perp is the projection of H onto M^\perp . We shall use the following elementary fact frequently:

Lemma 4.1. *Let K and L be Hilbert spaces, $M \subset K$ and $N \subset L$ be closed subspaces. Let $A : K \rightarrow L$ be a bounded operator. Assume that $A(M) \subset N$. Let $\tilde{A} : K/M \rightarrow L/N$ be the induced map such that $\tilde{A}([x]) = [Ax]$ for $x \in K$. Identifying K/M and L/N with M^\perp and N^\perp , \tilde{A} is identified with the bounded operator $S : M^\perp \rightarrow N^\perp$ such that $S(x) = P_N^\perp(Ax)$. Then $S = (A^*|_{N^\perp})^*$.*

Proof. Consider $A^* : L \rightarrow K$. Since $A(M) \subset N$, we have $A^*(N^\perp) \subset M^\perp$. Hence the restriction $A^*|_{N^\perp} : N^\perp \rightarrow M^\perp$ has the adjoint

$$(A^*|_{N^\perp})^* : M^\perp \rightarrow N^\perp.$$

For any $m \in M^\perp$ and $n \in N^\perp$

$$((A^*|_{N^\perp})^* m | n) = (m | A^*|_{N^\perp} n) = (m | A^* n) = (Am | n) = (P_N^\perp(Am) | n).$$

□

Definition. (reflection functor Φ_v^- .) Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we define a *reflection functor* at v

$$\Phi_v^- : H\text{Rep}(\Gamma) \rightarrow H\text{Rep}(\sigma_v^-(\Gamma))$$

between the categories of Hilbert representations of Γ and $\sigma_v^-(\Gamma)$ as follows: For a Hilbert representation (H, f) of Γ , we shall define a Hilbert representation $(K, g) = \Phi_v^-(H, f)$ of $\sigma_v^-(\Gamma)$. Let

$$\hat{h}_v : H_v \rightarrow \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$$

be a bounded linear operator defined by

$$\hat{h}_v(x) = (f_\alpha(x))_{\alpha \in E_v} \text{ for } x \in H_v.$$

Define

$$K_v := (\text{Im } \hat{h}_v)^\perp = \text{Ker } \hat{h}_v^* \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)},$$

where $\hat{h}_v^* : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \rightarrow H_v$ is given $\hat{h}_v^*((x_\alpha)_{\alpha \in E_v}) = \sum f_\alpha^*(x_\alpha)$. For $u \in E$ with $u \neq v$, put $K_u = H_u$.

Let $Q_v : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \rightarrow K_v$ be the canonical projection. For $\beta \in E_v$, let

$$j_\beta : H_{r(\beta)} \rightarrow \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$$

be the canonical inclusion. Define

$$g_\beta : K_{s(\beta)} = H_{r(\beta)} \rightarrow K_{r(\beta)} = K_v \text{ by } g_\beta = Q_v \circ j_\beta.$$

For $\beta \notin E_v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H, f) \rightarrow (H', f')$, we shall define a homomorphism

$$S = (S_u)_{u \in V} = \Phi_v^-(T) : (K, g) = \Phi_v^-(H, f) \rightarrow (K', g') = \Phi_v^-(H', f'),$$

recalling the above Lemma 4.1. For $u = v$, a bounded operator $S_v : K_v \rightarrow K'_v$ is given by

$$S_v((x_\alpha)_{\alpha \in E_v}) = Q'_v((T_{r(\alpha)}(x_\alpha))_{\alpha \in E_v}),$$

where $Q'_v : \bigoplus_{\alpha \in E_v} H'_{r(\alpha)} \rightarrow K'_v$ be the canonical projection.

We have the following commutative diagram:

$$\begin{array}{ccccccc} H_v & \xrightarrow{\hat{h}_v} & \bigoplus_{\alpha \in E_v} H_{r(\alpha)} & \xrightarrow{Q_v} & K_v & \longrightarrow & 0 \\ T_v \downarrow & & \bigoplus_{\alpha \in E_v} T_{r(\alpha)} \downarrow & & S_v \downarrow & & \\ H'_v & \xrightarrow{\hat{h}'_v} & \bigoplus_{\alpha \in E_v} H'_{r(\alpha)} & \xrightarrow{Q'_v} & K'_v & \longrightarrow & 0 \end{array}$$

For other $u \in V$ with $u \neq v$, we put

$$S_u = T_u : K_u = H_u \rightarrow K'_u = H'_u.$$

We shall explain a relation between two (covariant) functors Φ_v^+ and Φ_v^- . We need to introduce another (contravariant) functor Φ^* in the first place.

Let $\Gamma = (V, E, s, r)$ be a finite quiver. We define the opposite quiver $\bar{\Gamma} = (\bar{V}, \bar{E}, s, r)$ by reversing all the arrows, that is,

$$\bar{V} = V \quad \text{and} \quad \bar{E} = \{\bar{\alpha} \mid \alpha \in E\}.$$

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver and $\bar{\Gamma} = (\bar{V}, \bar{E}, s, r)$ its opposite quiver. We introduce a contravariant functor

$$\Phi^* : HRep(\Gamma) \rightarrow HRep(\bar{\Gamma})$$

between the categories of Hilbert representations of Γ and $\bar{\Gamma}$ as follows: For a Hilbert representation (H, f) of Γ , we shall define a Hilbert representation $(K, g) = \Phi^*(H, f)$ of $\bar{\Gamma}$ by

$$K_u = H_u \text{ for } u \in V \quad \text{and} \quad g_{\bar{\alpha}} = f_{\alpha}^* \text{ for } \alpha \in E.$$

For a homomorphism $T : (H, f) \rightarrow (H', f')$, we shall define a homomorphism

$$S = (S_u)_{u \in V} = \Phi^*(T) : (K', g') = \Phi^*(H', f') \rightarrow (K, g) = \Phi^*(H, f),$$

by bounded operators $S_u : K'_u = H'_u \rightarrow K_u = H_u$ given by $S_u = T_u^*$.

Proposition 4.2. *Let $\Gamma = (V, E, s, r)$ be a finite quiver. If $v \in V$ is a source of Γ , then v is a sink of $\bar{\Gamma}$, $\sigma_v^-(\Gamma) = \overline{\sigma_v^+(\bar{\Gamma})}$ and we have the following:*

(1) *For a Hilbert representation (H, f) of Γ ,*

$$\Phi_v^-(H, f) = \Phi^*(\Phi_v^+(\Phi^*(H, f))).$$

(2) *For a homomorphism $T : (H, f) \rightarrow (H', f')$,*

$$\Phi_v^-(T) = \Phi^*(\Phi_v^+(\Phi^*(T))).$$

Proof. (1): It is enough to consider around a source v . For each $\alpha \in E_v$ with $\alpha : v \rightarrow u = r(\alpha)$, a bounded operator $f_{\alpha} : H_v \rightarrow H_u$ is assigned in (H, f) . Taking Φ^* , we have $\Phi^*(H_u) = H_u$ and $\Phi^*(f_{\alpha}) = f_{\alpha}^* : H_u \rightarrow H_v$ in $\Phi^*(H, f)$. Let

$$h_v : \oplus_{\alpha \in E_v} H_{r(\alpha)} \rightarrow H_v$$

be a bounded operator given by

$$h_v((x_{\alpha})_{\alpha \in E_v}) = \sum_{\alpha \in E_v} f_{\alpha}^*(x_{\alpha}).$$

Define

$$W_v := \{(x_{\alpha})_{\alpha \in E_v} \in \oplus_{\alpha \in E_v} H_{r(\alpha)} \mid \sum_{\alpha \in E_v} f_{\alpha}^*(x_{\alpha}) = 0\}.$$

Then $\Phi_v^+(\Phi^*(H_v)) = W_v$ and $\Phi_v^+(\Phi^*(H_u)) = H_u$ in $\Phi^*(\Phi^*(H, f))$. Consider the canonical inclusion map $i_v : W_v \rightarrow \oplus_{\alpha \in E_v} H_{r(\alpha)}$. For $\beta \in E_v$, let

$$P_{\beta} : \oplus_{\alpha \in E_v} H_{r(\alpha)} \rightarrow H_{r(\beta)}$$

be the canonical projection. Then $\Phi_v^+(\Phi^*(f_\beta)) = P_\beta \circ i_v$. Finally take Φ^* again. Since $h_v^* : H_v \rightarrow \oplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by

$$(h_v^*)(y) = (f_\alpha(y))_{\alpha \in E_v} = \hat{h}_v(y), \quad \text{for } y \in H_v.$$

we have

$$\Phi^*(\Phi_v^+(\Phi^*(H_v))) = W_v = \text{Ker } h_v = (\text{Im } h_v^*)^\perp = (\text{Im } \hat{h}_v)^\perp = \Phi_v^-(H_v).$$

Moreover $i_v^* = Q_v : \oplus_{\alpha \in E_v} H_{r(\alpha)} \rightarrow W_v$ is the canonical projection. For $\beta \in E_v$, we have

$$P_\beta^* = j_\beta : H_{r(\beta)} \rightarrow \oplus_{\alpha \in E_v} H_{r(\alpha)}.$$

Therefore

$$\Phi^*(\Phi_v^+(\Phi^*(f_\beta))) = (P_\beta \circ i_v)^* = i_v^* \circ P_\beta^* = Q_v \circ j_\beta = \Phi_v^-(f_\beta).$$

(2): If $u \neq v$, then

$$(\Phi^*(\Phi_v^+(\Phi^*(T))))_u = T_u^{**} = T_u = (\Phi_v^-(T))_u.$$

If $u = v$, then, apply Lemma 4.1 by putting that $K = \oplus_{\alpha \in E_v} H_{r(\alpha)}$, $L = \oplus_{\alpha \in E_v} H'_{r(\alpha)}$, M is the closure of $\{(f_\alpha(x))_{\alpha \in E_v} \in K \mid x \in H_v\}$ in K , N is the closure of $\{(f'_\alpha(x))_{\alpha \in E_v} \in L \mid x \in H'_v\}$ in L and $A : K \rightarrow L$ with $A((y_\alpha)_{\alpha \in E_v}) = (T_{r(\alpha)}y_\alpha)_{\alpha \in E_v}$. Then $(\Phi^*(\Phi_v^+(\Phi^*(T))))_v = (\Phi_v^-(T))_v$. \square

Proposition 4.3. *Let $\Gamma = (V, E, s, r)$ be a finite quiver. If $v \in V$ is a sink of Γ , then v is a source of $\bar{\Gamma}$, $\sigma_v^+(\Gamma) = \overline{\sigma_v^-(\bar{\Gamma})}$ and we have the following:*

(1) *For a Hilbert representation (H, f) of Γ ,*

$$\Phi_v^+(H, f) = \Phi^*(\Phi_v^-(\Phi^*(H, f))).$$

(2) *For a homomorphism $T : (H, f) \rightarrow (H', f')$,*

$$\Phi_v^+(T) = \Phi^*(\Phi_v^-(\Phi^*(T))).$$

Proof. It follows immediately from Proposition 4.2 and the fact that $(\Phi^*)^2 = \text{Id}$. \square

5. DUALITY THEOREM

We shall show a certain duality between reflection functors. Bernstein-Gelfand-Ponomarev [BGP] introduced reflection functors and Coxeter functors and clarify a relation with the Coxeter-Weyl group and Dynkin diagrams in the case of finite-dimensional representations of quivers. In the case of infinite-dimensional Hilbert representations, duality theorem between reflection functors does not hold as in the purely algebraic setting. We need to modify and assume a certain closedness condition at a sink or a source.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Recall that $E^v = \{\alpha \mid r(\alpha) = v\}$. We say that a Hilbert representation

(H, f) of Γ is *closed* at v if $\sum_{\alpha \in E^v} \text{Im } f_\alpha \subset H_v$ is a closed subspace. We say that (H, f) is *full* at v if $\sum_{\alpha \in E^v} \text{Im } f_\alpha = H_v$.

Remark. Recall that a bounded operator $h_v : \oplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v$ is given by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$. Then a Hilbert representation (H, f) of Γ is *closed* at v if and only if $\text{Im } h_v$ is closed. A Hilbert representation (H, f) is *full* at v if and only if h_v is onto.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Recall that $E_v = \{\alpha | s(\alpha) = v\}$. We say that a Hilbert representation (H, f) of Γ is *co-closed* at v if $\sum_{\alpha \in E_v} \text{Im } f_\alpha^* \subset H_v$ is a closed subspace. We say that (H, f) is *co-full* at v if $\sum_{\alpha \in E_v} \text{Im } f_\alpha^* = H_v$.

Remark. Recall that a bounded operator $\hat{h}_v : H_v \rightarrow \oplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by $\hat{h}_v(x) = (f_\alpha(x))_{\alpha \in E_v}$ for $x \in H_v$. Then a Hilbert representation (H, f) of Γ is co-closed at v if and only if $\text{Im } \hat{h}_v^*$ is closed. A Hilbert representation (H, f) is co-full at v if and only if \hat{h}_v^* is onto if and only if $\text{Im } \hat{h}_v$ is closed and $\cap_{\alpha \in E_v} \text{Ker } f_\alpha = 0$. In fact the latter condition is equivalent to $(\text{Im } \hat{h}_v^*)^\perp = \text{Ker } \hat{h}_v = 0$. We also see that (H, f) is co-closed at v if and only if $\Phi_v^*(H, f)$ is closed at v . And (H, f) is co-full at v if and only if $\Phi_v^*(H, f)$ is full at v .

In order to prove a duality theorem, we need to prepare a lemma.

Lemma 5.1. *Let H and K be Hilbert spaces and $T : H \rightarrow K$ be a bounded operator. Let $T = U|T|$ be its polar decomposition and U a partial isometry with $\text{supp } U = \overline{\text{Im } |T|}$ and $\text{Im } U = \overline{\text{Im } T}$. Suppose that $\text{Im } T$ is closed. Then we have the following:*

- (1) $\text{Im } |T| = \text{Im } T^*$ is a closed subspace of H .
- (2) Under the orthogonal decomposition

$$H = \text{Ker } |T|^\perp \oplus \text{Ker } |T| = \text{Im } |T| \oplus \text{Ker } |T|,$$

the restriction $|T|_{|\text{Im } |T|} : \text{Im } |T| \rightarrow \text{Im } |T|$ is a bounded invertible operator.

- (3) Let $S = (|T|_{|\text{Im } |T|})^{-1}$ be its inverse. Define a bounded operator $B : K \rightarrow \text{Im } T^*$ by $Bx = SU^*x$ for $x \in K$. Let $Q : H \rightarrow \text{Im } T^*$ be the canonical projection. Then $BT = Q$. Moreover $B|_{\text{Im } T} : \text{Im } T \rightarrow \text{Im } T^*$ is a bounded invertible operator.

Proof. (1) Since $\text{Im } T$ is closed, $\text{Im } T^*$ is also closed. Since $U(|T|x) = Tx$ by definition of U and $\text{Im } T$ is closed, $\text{Im } |T|$ is closed.

(2) Since $\text{Ker } |T|^\perp = \text{Im } |T|$, $|T|_{|\text{Im } |T|}$ is one to one. Since $|T|(H) = |T|(\text{Im } |T|)$ is closed, $|T|_{|\text{Im } |T|}$ is onto. Hence $|T|_{|\text{Im } |T|}$ is bounded invertible.

(3) For any $x = x_1 + x_2 \in H$ with $x_1 \in \text{Im } |T| = \text{Im } T^*$ and $x_2 \in \text{Ker } |T|$,

$$BTx = SU^*U|T|x = S|T|x = S|T|x_1 = x_1 = Qx.$$

It is clear that $B|_{\text{Im } T}$ is a bounded invertible operator. \square

Theorem 5.2. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Assume that a Hilbert representation (H, f) of Γ is closed at v . Let $h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v$ be a bounded operator defined by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$. Define a Hilbert representation (\tilde{H}, \tilde{f}) of Γ by $\tilde{H}_v = (\text{Im } h_v)^\perp \subset H_v$, $\tilde{H}_u = 0$ for $u \neq v$ and $\tilde{f} = 0$. Then we have*

$$(H, f) \cong \Phi_v^-(\Phi_v^+(H, f)) \oplus (\tilde{H}, \tilde{f}).$$

Proof. Let $(H^+, f^+) = \Phi_v^+(H, f)$ and $(H^{+-}, f^{+-}) = \Phi_v^-(\Phi_v^+(H, f))$. Then $H_v^+ = \text{Ker } h_v = \{(x_\alpha)_{\alpha \in E^v} \in \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \mid \sum_{\alpha \in E^v} f_\alpha(x_\alpha) = 0\}$, and $H_u^+ = H_u$ for $u \neq v$. We have $f_\beta^+((x_\alpha)_{\alpha \in E^v}) = x_\beta$ for $\beta \in E^v$, and $f_\beta^+ = f_\beta$ for $\beta \notin E^v$.

Let $\hat{h}_v : H_v^+ \rightarrow \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ be a bounded operator given by

$$\hat{h}_v((x_\alpha)_{\alpha \in E^v}) = (f_\beta^+((x_\alpha)_{\alpha \in E^v}))_{\beta \in E^v} = (x_\beta)_{\beta \in E^v} = (x_\alpha)_{\alpha \in E^v}.$$

Hence \hat{h}_v is the canonical embedding. Since (H, f) is closed at v , $\text{Im } h_v$ and $\text{Im } h_v^*$ are closed subspaces. Therefore

$$H_v^{+-} = (\text{Im } \hat{h}_v)^\perp = (H_v^+)^\perp = (\text{Ker } h_v)^\perp = \text{Im } h_v^*.$$

For any other $u \in V$ with $u \neq v$, $H_u^{+-} = H_u$. Let $Q_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v^{+-}$ be the canonical projection. For $\beta \in E^v$, let

$$j_\beta : H_{s(\beta)} \rightarrow \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$$

be the canonical inclusion. Then $f_\beta^{+-} : H_{s(\beta)} \rightarrow H_v^{+-}$ is given by $f_\beta^{+-} = Q_v \circ j_\beta$. For other $\beta \notin E^v$, we have $f_\beta^{+-} = f_\beta$.

We shall define an isomorphism

$$\varphi : (H, f) \rightarrow \Phi_v^-(\Phi_v^+(H, f)) \oplus (\tilde{H}, \tilde{f}).$$

Apply Lemma 5.1 by putting $T = h_v$, $H = \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ and $K = H_v$. Consider the polar decomposition $h_v = U|h_v|$. Put $S = (|h_v|||_{\text{Im } h_v})^{-1}$. Define a bounded operator $B : H_v \rightarrow \text{Im } h_v^*$ by $B = SU^*$. Then Bh_v is the canonical projection Q_v of H_v onto $\text{Im } h_v^*$. We define

$$\varphi_v : H_v = \text{Im } h_v \oplus (\text{Im } h_v)^\perp \rightarrow H_v^{+-} \oplus \tilde{H}_v = \text{Im } h_v^* \oplus (\text{Im } h_v)^\perp$$

by $\varphi_v(x, y) = (B|_{\text{Im } h_v} x, y)$ for $x \in \text{Im } h_v$ and $y \in (\text{Im } h_v)^\perp$. By Lemma 5.1 (2), φ_v is a bounded invertible operator. For $u \in V$ with $u \neq v$, put $\varphi_u : H_u \rightarrow H_u \oplus 0$ by $\varphi_u(x) = (x, 0)$ for $x \in H_u$. For any $\beta \in E^v$ and $x \in H_{s(\beta)}$,

$$\varphi_v \circ f_\beta(x) = \varphi_v(h_v(j_\beta(x))) = (B(h_v(j_\beta(x))), 0) = (Q_v(j_\beta(x)), 0).$$

On the other hand,

$$(f_\beta^{+-} \oplus 0) \circ \varphi_{s(\beta)}(x) = (f_\beta^{+-} \oplus 0)(x, 0) = (f_\beta^{+-}(x), 0) = (Q_v \circ j_\beta(x), 0).$$

For other $\beta \notin E^v$, we have

$$\varphi_{r(\beta)} \circ f_\beta^{+-} = \varphi_{r(\beta)} \circ f_\beta = f_\beta \circ \varphi_{s(\beta)} = f_\beta^{+-} \circ \varphi_{s(\beta)}.$$

Hence $\varphi : (H, f) \rightarrow \Phi_v^-(\Phi_v^+(H, f)) \oplus (\tilde{H}, \tilde{f})$ is an isomorphism. \square

Counter example. If we do not assume that a Hilbert representation (H, f) of Γ is closed at v , then the above Theorem 5.2 does not hold in general. In fact, consider the following quiver $\Gamma = (V, E, s, r)$:

$$\circ_1 \xrightarrow{\alpha_1} \circ_0 \xleftarrow{\alpha_2} \circ_2$$

Let $K = \ell^2(\mathbb{N})$ with the canonical basis $(e_n)_{n \in \mathbb{N}}$. Define a Hilbert representation (H, f) of Γ by $H_0 = K \oplus K$, $H_1 = K \oplus 0$ and H_2 is the closed subspace of H_0 spanned by $\{(\cos \frac{\pi}{n} e_n, \sin \frac{\pi}{n} e_n) \in K \oplus K \mid n \in \mathbb{N}\}$. Then $H_1 \cap H_2 = 0$ and $H_1 + H_2$ is a dense subspace of H_0 but not closed in H_0 . Let $f_k = f_{\alpha_k} : H_k \rightarrow H_0$ be the inclusion map for $k = 1, 2$. Then (H, f) is not closed at a sink 0. It is easy to see that $H_0^+ = \text{Ker } h_0 = 0$, $f_1^+ = 0$ and $f_2^+ = 0$. Therefore $H_0^{+-} = H_1 \oplus H_2$ and $H_1^{+-} = H_1$, $H_2^{+-} = H_2$. We have $f_k^{+-} : H_k \rightarrow H_1 \oplus H_2$ is a canonical inclusion for $k = 1, 2$. Since $\tilde{H}_0 = (\text{Im } h_v)^\perp = 0$, we have $(\tilde{H}, \tilde{f}) = (0, 0)$. Therefore

$$\Phi_v^-(\Phi_v^+(H, f)) \oplus (\tilde{H}, \tilde{f}) = \Phi_v^-(\Phi_v^+(H, f)) = (H^{+-}, f^{+-})$$

is closed at a sink 0. But (H, f) is not closed at a sink 0. Therefore there exists no isomorphism between (H, f) and $\Phi_v^-(\Phi_v^+(H, f)) \oplus (\tilde{H}, \tilde{f})$.

Note that (H, f) is not full at a sink 0 and $\Phi_v^-(\Phi_v^+(H, f))$ is full at a sink 0. Therefore this example also shows that, if we do not assume that a Hilbert representation (H, f) of Γ is full at v , then the following Duality Theorem (Corollary 5.3) does not hold in general.

Corollary 5.3. (Duality theorem.) *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. If a Hilbert representation (H, f) of Γ is full at v , then*

$$(H, f) \cong \Phi_v^-(\Phi_v^+(H, f)).$$

Proof. Since (H, f) is full at v , $\tilde{H}_v = (\text{Im } h_v)^\perp = H_v^\perp = 0$. Hence $(\tilde{H}, \tilde{f}) = (0, 0)$ in Theorem 5.2. \square

Remark. It is also necessary that (H, f) is full at the sink v in order that the above Duality Theorem holds. It follows from Lemma 5.8 below.

We have a dual version.

Theorem 5.4. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Assume that a Hilbert representation (H, f) of Γ is co-closed at v . Let $\hat{h}_v : H_v \rightarrow \oplus_{\alpha \in E_v} H_{r(\alpha)}$ is a bounded operator defined by $\hat{h}_v(x) = (f_\alpha(x))_{\alpha \in E_v}$ for $x \in H_v$. Define a Hilbert representation (\check{H}, \check{f}) of Γ by*

$$\check{H}_v = (\text{Im } \hat{h}_v^*)^\perp (= \text{Ker } \hat{h}_v = \cap_{\alpha \in E_v} \text{Ker } f_\alpha) \subset H_v,$$

$\check{H}_u = 0$ for $u \neq v$ and $\check{f} = 0$. Then

$$(H, f) \cong \Phi_v^+(\Phi_v^-(H, f)) \oplus (\check{H}, \check{f}).$$

Proof. We see that v is a sink in $\bar{\Gamma}$, because v is a source in Γ . Since a Hilbert representation (H, f) of Γ is co-closed at v , a Hilbert representation $\Phi_v^*(H, f)$ is closed at v . By Theorem 5.2, there exists a Hilbert representation (\tilde{H}, \tilde{f}) of $\bar{\Gamma}$ such that

$$\Phi_v^*(H, f) \cong \Phi_v^-(\Phi_v^+(\Phi_v^*(H, f))) \oplus (\tilde{H}, \tilde{f}).$$

Put $(\check{H}, \check{f}) = \Phi_v^*(\tilde{H}, \tilde{f})$. Then

$$\begin{aligned} (H, f) &\cong \Phi_v^*(\Phi_v^*(H, f)) \cong \Phi_v^* \Phi_v^- \Phi_v^+ \Phi_v^*(H, f) \oplus \Phi_v^*(\tilde{H}, \tilde{f}) \\ &\cong (\Phi_v^* \Phi_v^- \Phi_v^*)(\Phi_v^* \Phi_v^+ \Phi_v^*)(H, f) \oplus \Phi_v^*(\tilde{H}, \tilde{f}) \\ &\cong \Phi_v^+(\Phi_v^-(H, f)) \oplus (\check{H}, \check{f}). \end{aligned}$$

Moreover it is easy to see that

$$\check{H}_v = \left(\sum_{\alpha \in E_v} \text{Im } f_\alpha^* \right)^\perp = \cap_{\alpha \in E_v} \text{Ker } f_\alpha.$$

□

Counter example. If we do not assume that a Hilbert representation (H, f) of Γ is co-closed at the source v , then the above Theorem 5.4 does not hold in general. In fact, consider the following quiver $\Gamma = (V, E, s, r)$:

$$\circ_1 \xleftarrow{\alpha_1} \circ_0 \xrightarrow{\alpha_2} \circ_2$$

Let $K = \ell^2(\mathbb{N})$ with the canonical basis $(e_n)_{n \in \mathbb{N}}$. Define a Hilbert representation (H, f) of Γ by $H_0 = K \oplus K$, $H_1 = K \oplus 0$ and H_2 is the closed subspace H_0 spanned by $\{(\cos \frac{\pi}{n} e_n, \sin \frac{\pi}{n} e_n) \in K \oplus K \mid n \in \mathbb{N}\}$. Let $f_k = f_{\alpha_k} : H_0 \rightarrow H_k$ be the canonical projection for $k = 1, 2$. Then (H, f) is not co-closed at a source 0. It is easy to see that $H_0^- = (\text{Im } \hat{h}_0)^\perp = 0$, $f_1^- = 0$ and $f_2^- = 0$. Therefore $H_0^{-+} = H_1 \oplus H_2$ and $H_1^{-+} = H_1$, $H_2^{-+} = H_2$. We have that $f_k^{-+} : H_1 \oplus H_2 \rightarrow H_k$ is the canonical projection for $k = 1, 2$. Since $\check{H}_0 = \text{Ker } \hat{h}_0 = 0$, we have $(\check{H}, \check{f}) = (0, 0)$. Therefore

$$\Phi_v^+(\Phi_v^-(H, f)) \oplus (\check{H}, \check{f}) = \Phi_v^+(\Phi_v^-(H, f)) = (H^{-+}, f^{-+})$$

is co-closed at a source 0. But (H, f) is not co-closed at a source 0. Therefore there exists no isomorphism between (H, f) and $\Phi_v^+(\Phi_v^-(H, f)) \oplus (\check{H}, \check{f})$.

Note that (H, f) is not co-full at a source 0 and $\Phi_v^+(\Phi_v^-(H, f))$ is co-full at a source 0. Therefore this example also shows that, if we do not assume that a Hilbert representation (H, f) of Γ is co-full at v , then the following Duality Theorem (Corollary 5.5) does not hold in general.

Corollary 5.5. (Duality theorem.) *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. If a Hilbert representation (H, f) of Γ is co-full at v , then*

$$(H, f) \cong \Phi_v^+(\Phi_v^-(H, f)).$$

Proof. Since (H, f) is co-full at v , $\check{H}_v = \bigcap_{\alpha \in E_v} \text{Ker } f_\alpha = 0$. Hence $(\check{H}, \check{f}) = (0, 0)$ in Theorem 5.4. \square

Remark. It is also necessary that (H, f) is co-full at the source v in order that the above Duality Theorem holds. It follows from Lemma 5.6 below.

Lemma 5.6. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Then for any Hilbert representation (H, f) of Γ , $\Phi_v^+(H, f)$ is co-full at v .*

Proof. Put $(H^+, f^+) = \Phi_v^+(H, f)$. Recall that $h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v$ is given by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$, and $H_v^+ = \text{Ker } h_v$. And For $\beta \in E^v$, let $i_v : H_v^+ \rightarrow \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ be the canonical inclusion and $P_\beta : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_{s(\beta)}$ the canonical projection. We define

$$f_{\bar{\beta}}^+ : H_{s(\bar{\beta})}^+ = H_v^+ \rightarrow H_{r(\bar{\beta})}^+ = H_{s(\beta)} \quad \text{by } g_{\bar{\beta}} = P_\beta \circ i_v.$$

Therefore $f_{\bar{\beta}}^{+*} : H_{s(\beta)} \rightarrow H_v^+$ is given by $f_{\bar{\beta}}^{+*} = i_v^* \circ P_\beta^*$. Since $P_\beta^* : H_{s(\beta)} \rightarrow \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ is the canonical inclusion and $i_v^* : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v^+$ is the canonical projection, we have

$$\sum_{\bar{\beta} \in E_v} \text{Im } f_{\bar{\beta}}^{+*} = \sum_{\beta \in E^v} \text{Im}(i_v^* \circ P_\beta^*) = H_v^+.$$

Therefore (H^+, f^+) is co-full at v . \square

Proposition 5.7. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. If (H, f) is a Hilbert representation of Γ , then*

$$\Phi_v^+ \Phi_v^- \Phi_v^+(H, f) \cong \Phi_v^+(H, f).$$

Proof. Since $\Phi_v^+(H, f)$ is co-full at the source v in $\sigma_v^+(\Gamma)$ by the above lemma 5.6, duality theorem (Corollary 5.5) yields the conclusion. \square

Lemma 5.8. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Then for any Hilbert representation (H, f) of Γ , $\Phi_v^-(H, f)$ is full at v .*

Proof. Put $(H^-, f^-) = \Phi_v^-(H, f)$. Recall that $\hat{h}_v : H_v \rightarrow \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by $\hat{h}_v(x) = (f_\alpha(x))_{\alpha \in E_v}$ for $x \in H_v$. and $H_v^- = (\text{Im } \hat{h}_v)^\perp \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$. Let $Q_v : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \rightarrow H_v^-$ be the canonical projection. For $\beta \in E_v$, let $j_\beta : H_{r(\beta)} \rightarrow \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ be the canonical inclusion. Then

$$f_{\bar{\beta}}^- : H_{s(\bar{\beta})}^- = H_{r(\beta)} \rightarrow H_{r(\bar{\beta})}^- = H_v^- \quad \text{by } f_{\bar{\beta}}^- = Q_v \circ j_\beta.$$

Therefore

$$\sum_{\bar{\beta} \in E^v} \text{Im } f_{\bar{\beta}}^- = Q_v(\oplus_{\alpha \in E^v} H_{r(\alpha)}) = H_v^-.$$

Thus (H^-, f^-) is full at v . \square

Proposition 5.9. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. If (H, f) is a Hilbert representation of Γ , then*

$$\Phi_v^- \Phi_v^+ \Phi_v^-(H, f) \cong \Phi_v^-(H, f).$$

Proof. Since $\Phi_v^-(H, f)$ is full at the source in $\sigma_v^-(\Gamma)$ by the above lemma 5.8, duality theorem (Corollary 5.3) yields the conclusion. \square

We examine on which representation a reflection functor vanishes.

Lemma 5.10. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Then, for any Hilbert representation (H, f) of Γ , the following are equivalent:*

- (1) $\Phi_v^+(H, f) \cong (0, 0)$
- (2) $H_u = 0$ for any $u \in V$ with $u \neq v$.

Furthermore if the above conditions are satisfied and (H, f) is indecomposable, then $H_v \cong \mathbb{C}$. If the above conditions are satisfied and (H, f) is full at the sink v , then $(H, f) \cong (0, 0)$.

Proof. Put $(H^+, f^+) = \Phi_v^+(H, f)$. Recall that $h_v : \oplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v$ is given by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$, and $H_v^+ = \text{Ker } h_v$. For other $u \in V$ with $u \neq v$, $H_u^+ = H_u$.

(1) \Rightarrow (2): Assume that $\Phi_v^+(H, f) = 0$. Then, for any $u \in V$ with $u \neq v$ we have $H_u = H_u^+ = 0$.

(2) \Rightarrow (1): Assume that $H_u = 0$ for any $u \in V$ with $u \neq v$. Then $H_v^+ = 0$, because $H_v^+ = \text{Ker } h_v \subset \oplus_{\alpha \in E^v} H_{s(\alpha)} = 0$. For other $u \in V$ with $u \neq v$, $H_u^+ = H_u = 0$.

Furthermore assume that the above conditions are satisfied and (H, f) is indecomposable. Then $f = 0$. Suppose that $\dim H_v \geq 2$. Then a non-trivial decomposition $H_v = K \oplus L$ gives a non-trivial decomposition of (H, f) . This contradicts that (H, f) is indecomposable. Hence $H_v \cong \mathbb{C}$. Assume that the above conditions are satisfied and (H, f) is full at v . Then $f = 0$, so that $H_v = \sum_{\alpha \in E^v} \text{Im } f_\alpha = 0$. Hence $(H, f) \cong (0, 0)$. \square

Lemma 5.11. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Then, for any Hilbert representation (H, f) of Γ , the following condition are equivalent:*

- (1) $\Phi_v^-(H, f) \cong (0, 0)$
- (2) $H_u = 0$ for any $u \in V$ with $u \neq v$.

Furthermore if the above conditions are satisfied and (H, f) is indecomposable, then $H_v \cong \mathbb{C}$. If the above conditions are satisfied and (H, f) is co-full at the source v , then $(H, f) \cong (0, 0)$.

Proof. Put $(H^-, f^-) = \Phi_v^-(H, f)$. Recall that $\hat{h}_v : H_v \rightarrow \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by $\hat{h}_v(x) = (f_\alpha(x))_{\alpha \in E_v}$ for $x \in H_v$, and $H_v^- = (\text{Im } \hat{h}_v)^\perp \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$. For other $u \in V$ with $u \neq v$, $H_u^- = H_u$.

(1) \Rightarrow (2): Assume that $\Phi_v^-(H, f) = 0$. Then, for any $u \in V$ with $u \neq v$ we have $H_u = H_u^- = 0$.

(2) \Rightarrow (1): Assume that $H_u = 0$ for any $u \in V$ with $u \neq v$. Then $H_v^- = 0$, because $H_v^- = (\text{Im } \hat{h}_v)^\perp \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)} = 0$. For other $u \in V$ with $u \neq v$, $H_u^- = H_u = 0$.

Assume that the above conditions are satisfied and (H, f) is co-full at v . Since $f_\alpha^* = 0$ for any $\alpha \in E$, $H_v = \sum_{\alpha \in E_v} \text{Im } f_\alpha^* = 0$. Hence $(H, f) \cong (0, 0)$. The rest is clear. \square

We shall show that a reflection functor preserves indecomposability of a Hilbert representation unless vanishing on it, under the assumption that the Hilbert representation is closed (resp. co-closed) at a sink (resp. source).

Theorem 5.12. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Suppose that a Hilbert representation (H, f) of Γ is indecomposable and closed at v . Then we have the following:*

- (1) *If $\Phi_v^+(H, f) = 0$, then $H_v = \mathbb{C}$, $H_u = 0$ for any $u \in V$ with $u \neq v$ and $f_\alpha = 0$ for any $\alpha \in E$.*
- (2) *If $\Phi_v^+(H, f) \neq 0$, then $\Phi_v^+(H, f)$ is also indecomposable and $(H, f) \cong \Phi_v^-(\Phi_v^+(H, f))$.*

Proof. Recall an operator $h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \rightarrow H_v$ defined by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$. Since (H, f) is closed at a sink v , we have a decomposition such that

$$(H, f) \cong \Phi_v^-(\Phi_v^+(H, f)) \oplus (\tilde{H}, \tilde{f})$$

by Theorem 5.2, where $\tilde{H}_v = (\text{Im } h_v)^\perp \subset H_v$, $\tilde{H}_u = 0$ for $u \neq v$ and $\tilde{f} = 0$.

Since (H, f) is indecomposable, $\Phi_v^-(\Phi_v^+(H, f)) \cong (0, 0)$ or $(\tilde{H}, \tilde{f}) \cong (0, 0)$.

(Case 1): Suppose that $\Phi_v^-(\Phi_v^+(H, f)) \cong (0, 0)$. Then $(H, f) \cong (\tilde{H}, \tilde{f})$. Hence $H_u \cong \tilde{H}_u = 0$ for $u \neq v$. This implies that $\Phi_v^+(H, f) \cong (0, 0)$ by Lemma 5.10. Since (H, f) is indecomposable, $H_v \cong \mathbb{C}$.

(Case 2): Suppose that $(\tilde{H}, \tilde{f}) \cong (0, 0)$. Then $(H, f) \cong \Phi_v^-(\Phi_v^+(H, f))$. Since (H, f) is non-zero, $\Phi_v^+(H, f)$ is non-zero. We shall show that $\Phi_v^+(H, f)$ is indecomposable. Assume that $\Phi_v^+(H, f) \cong (K, g) \oplus (K', g')$. Then

$$(H, f) \cong \Phi_v^-(\Phi_v^+(H, f)) \cong \Phi_v^-(K, g) \oplus \Phi_v^-(K', g').$$

Since (H, f) is indecomposable, $\Phi_v^-(K, g) \cong (0, 0)$ or $\Phi_v^-(K', g') \cong (0, 0)$. By Lemma 5.6, $\Phi_v^+(H, f)$ is co-full at v , so are its direct summands (K, g) and (K', g') . Then $(K, g) \cong (0, 0)$ or $(K', g') \cong (0, 0)$ by Lemma 5.11. Thus $\Phi_v^+(H, f)$ is indecomposable.

Since (Case 1) and (Case 2) are mutually exclusive and either of them occurs, we get the conclusion. \square

We have a dual version.

Theorem 5.13. *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Suppose that a Hilbert representation (H, f) of Γ is indecomposable and co-closed at v . Then we have the following:*

- (1) *If $\Phi_v^-(H, f) = 0$, then $H_v = \mathbb{C}$, $H_u = 0$ for any $u \in V$ with $u \neq v$ and $f_\alpha = 0$ for any $\alpha \in E$.*
- (2) *If $\Phi_v^-(H, f) \neq 0$, then $\Phi_v^-(H, f)$ is also indecomposable and $(H, f) \cong \Phi_v^+ \Phi_v^-(H, f)$.*

Proof. A dual argument of the proof in Theorem 5.12 works. \square

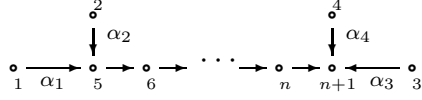
6. EXTENDED DYNKIN DIAGRAMS

Gabriel's theorem says that a connected finite quiver has only finitely many indecomposable representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8 . Representations of quivers on finite-dimensional vector spaces has been developed by Bernstein-Gelfand-Ponomarev [BGP], Donovan-Freislich [DF], V. Dlab-Ringel [DR], Gabriel-Roiter [GR], Kac [Ka], Nazarova [Na]

Furthermore locally scalar representations of quivers in the category of Hilbert spaces up to the unitary equivalence were introduced by Kruglyak and Roiter [KR]. They prove an analog of Gabriel's theorem.

We consider a complement of Gabriel's theorem for Hilbert representations. We need to construct some examples of indecomposable, infinite-dimensional representations of quivers with the underlying undirected graphs extended Dynkin diagrams \tilde{D}_n ($n \geq 4$), \tilde{E}_7 and \tilde{E}_8 . We consider the relative position of several subspaces along the quivers, where vertices are represented by a family of subspaces and arrows are represented by natural inclusion maps.

Lemma 6.1. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram \tilde{D}_n for $n \geq 4$:*

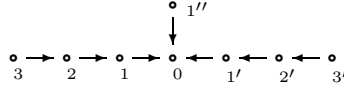


Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ .

Proof. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K . We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows: Define $H_1 = K \oplus 0$, $H_2 = 0 \oplus K$, $H_3 = \{(x, Sx) \in K \oplus K | x \in K\}$, $H_4 = \{(x, x) \in K \oplus K | x \in K\}$. $H_5 = H_6 = \dots = H_{n+1} = K \oplus K$, Let $f_{\alpha_k} : H_{s(\alpha_k)} \rightarrow H_{r(\alpha_k)}$ be the inclusion map for any $\alpha_k \in E$ for $k = 1, 2, 3, 4$, and $f_\beta = id$ for other arrows $\beta \in E$. Then we can show that (H, f) is indecomposable as in Example 3 in section 3. \square

Let $\Gamma = (V, E, s, r)$ be the quiver of Example 4 in section 3. with the underlying undirected graph a extended Dynkin diagram \tilde{E}_6 . We have already shown that there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ .

Lemma 6.2. Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram \tilde{E}_7 :



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ .

Proof. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K . We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows: Let $H_0 = K \oplus K \oplus K \oplus K$, $H_1 = K \oplus 0 \oplus K \oplus K$, $H_2 = K \oplus 0 \oplus \{(x, x); x \in K\}$, $H_3 = K \oplus 0 \oplus 0 \oplus 0$, $H_{1'} = 0 \oplus K \oplus K \oplus K$, $H_{2'} = 0 \oplus K \oplus \{(y, Sy) \in K^2 | y \in K\}$, $H_{3'} = 0 \oplus K \oplus 0 \oplus 0$ and $H_{1''} = \{(x, y, x, y) \in K^4 | x, y \in K\}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \rightarrow H_{r(\alpha)}$ be the canonical inclusion map. We shall show that the Hilbert representation (H, f) is indecomposable. Take $T = (T_v)_{v \in V} \in Idem(H, f)$. Since $T \in End(H, f)$ and any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in \{1, 2, 3, 1', 2', 3', 1''\}$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Since T_0 preserves $H_3 = K \oplus 0 \oplus 0 \oplus 0$, $H_{3'} = 0 \oplus K \oplus 0 \oplus 0$, and $H_{1'} \cap H_1 = 0 \oplus 0 \oplus K \oplus K$, T_0 is written

$$T_0 = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & X & Y \\ 0 & 0 & Z & W \end{pmatrix},$$

23

for some $A, B, X, T, Z, W \in B(K)$.

Because $H_{1''} = \{(x, y, x, y) \in K^4 \mid x, y \in K\}$ is also invariant under T_0 , for any $x, y \in K$, there exist $x', y' \in K$ such that

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & X & Y \\ 0 & 0 & Z & W \end{pmatrix} \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ By \\ Xx + Yy \\ Zx + Wy \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ x' \\ y' \end{pmatrix}.$$

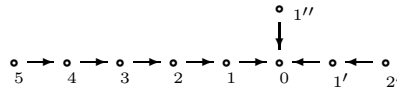
Putting $y = 0$, we have $Ax = Xx$ and $0 = Zx$ for any $x \in K$. Hence $A = X$ and $Z = 0$. Similarly, letting $x = 0$, we have $Y = 0$ and $W = B$. Therefore T_0 has a block diagonal form such that

$$T_0 = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{pmatrix} = A \oplus B \oplus A \oplus B.$$

Furthermore, as T_0 preserves $H_{1'} \cap H_2 = \{(0, 0, x, x) \in K^4 \mid x \in K\}$, for any $x \in K$ there exists $y \in K$ such that $(0, 0, Ax, Bx) = (0, 0, y, y)$. Hence $A = B$. Therefore $T_0 = A \oplus A \oplus A \oplus A$. Moreover $H_1 \cap H_{2'} = \{(0, 0, x, Sx) \in K^4 \mid x \in K\}$ is also invariant under T_0 . Hence for any $x \in K$, there exists $y \in K$ such that $(0, 0, Ax, ASx) = (0, 0, y, Sy)$. Thus $AS = SA$. Since $T \in \text{Idem}(H, f)$, T_0 is idempotent, so that A is also idempotent. Because $AS = SA$ and $A^2 = A$, we have $A = 0$ or $A = I$. Thus $T_0 = 0$ or $T_0 = I$. Since for any $v \in V$ and any $x \in H_v$ $T_0x = T_vx$, we have $T_v = 0$ or $T_v = I$ simultaneously. Thus $T = (T_v)_{v \in V} = 0$ or $T = I$, that is, $\text{Idem}(H, f) = \{0, I\}$. Therefore (H, f) is indecomposable. \square

Remark. Replacing S by $S + \lambda I$ for $\lambda \in \mathbb{C}$, we have uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ .

Lemma 6.3. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram \tilde{E}_8 :*



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ .

Proof. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K . We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:

Let $H_0 = K \oplus K \oplus K \oplus K \oplus K \oplus K$,

$H_1 = \{(x, x) \in K^2 \mid x \in K\} \oplus K \oplus K \oplus K \oplus K$,

$H_2 = 0 \oplus 0 \oplus K \oplus K \oplus K \oplus K$, $H_3 = 0 \oplus 0 \oplus 0 \oplus K \oplus K \oplus K$,

$H_4 = 0 \oplus 0 \oplus 0 \oplus K \oplus \{(y, Sy) \in K^2 \mid y \in K\}$, $H_5 = 0 \oplus 0 \oplus 0 \oplus K \oplus 0 \oplus 0$,

$H_{1'} = K \oplus K \oplus \{(x, y, x, y) \in K^4 \mid x, y \in K\}$, $H_{2'} = K \oplus K \oplus 0 \oplus 0 \oplus 0 \oplus 0$,

$$H_{1''} = \{(y, z, x, 0, y, z) \in K^6 \mid x, y, z \in K\}.$$

For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \rightarrow H_{r(\alpha)}$ be the canonical inclusion map. We shall show that the Hilbert representation (H, f) is indecomposable. Take $T = (T_v)_{v \in V} \in \text{Idem}(H, f)$. Since $T \in \text{End}(H, f)$ and any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in V$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Since T_0 preserves subspaces $H_{2'} = K \oplus K \oplus 0 \oplus 0 \oplus 0 \oplus 0$, $H_2 = 0 \oplus 0 \oplus K \oplus K \oplus K \oplus K$, T_0 has a form such that

$$T_0 = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

for some $A \in B(K \oplus K)$ and $B \in B(K \oplus K \oplus K \oplus K)$.

Moreover $H_{1''} \cap H_2 = 0 \oplus 0 \oplus K \oplus 0 \oplus 0 \oplus 0$ and $H_3 = 0 \oplus 0 \oplus 0 \oplus K \oplus K \oplus K$, are invariant under T_0 . Furthermore $H_5 = 0 \oplus 0 \oplus 0 \oplus K \oplus 0 \oplus 0$ and $T_0(H_5) \subset H_5$. Therefore T_0 is written as

$$T_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & f & g & h \\ 0 & 0 & 0 & 0 & i & j \\ 0 & 0 & 0 & 0 & k & l \end{pmatrix},$$

for some $a, b, c, d, e, f, g, h, i, j, k, l \in B(K)$.

Since $H_{1'} \cap H_3 = 0 \oplus 0 \oplus 0 \oplus \{(y, 0, y) \in K^4 \mid y \in K\}$ is invariant under T_0 , for any $y \in K$, there exists $y' \in K$ such that

$$B \begin{pmatrix} 0 \\ y \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & l \end{pmatrix} \begin{pmatrix} 0 \\ y \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ fy + hy \\ jy \\ ly \end{pmatrix} = \begin{pmatrix} 0 \\ y' \\ 0 \\ y' \end{pmatrix}.$$

Therefore $f + h = l$ and $j = 0$. Next consider $H_{1'} \cap H_2 = 0 \oplus 0 \oplus \{(x, y, x, y); x, y \in K\}$. Since $H_{1'} \cap H_2$ is invariant under T_0 , for any $x, y \in K$ there exist $x', y' \in K$ such that

$$B \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & f & g & h \\ 0 & 0 & i & 0 \\ 0 & 0 & k & l \end{pmatrix} \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = \begin{pmatrix} ex \\ fy + gx + hy \\ ix \\ kx + ly \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ x' \\ y' \end{pmatrix}$$

Putting $y = 0$, we have

$$ex = x' = ix, \quad gx = y' = kx \quad \text{for any } x \in K.$$

Hence $e = i$ and $g = k$.

Letting $x = 0$, we have $fy + hy = y' = ly$ for any $y \in K$. Hence $f + h = l$.

Since T_0 preserves $H_{2'} \cap H_1 = \{(x, x) \in K^2 \mid x \in K\} \oplus 0 \oplus 0 \oplus 0 \oplus 0$, for any $x \in K$, there exists $x' \in K$ such that

$$A \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} ax + bx \\ cx + dx \end{pmatrix} = \begin{pmatrix} x' \\ x' \end{pmatrix}.$$

Hence $ax + bx = cx + dx$, for any $x \in K$, so that $a + b = c + d$.

Furthermore $H_{1''} = \{(y, z, x, 0, y, z) \in K^6 \mid x, y, z \in K\}$ is invariant under T_0 . Therefore for any $x, y, z \in K$ there exist $x', y', z' \in K$ satisfying

$$\begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & f & g & h \\ 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & g & l \end{pmatrix} \begin{pmatrix} y \\ z \\ x \\ 0 \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + bz \\ cy + dz \\ ex \\ gy + hz \\ ey \\ gy + lz \end{pmatrix} = \begin{pmatrix} y' \\ z' \\ x' \\ 0 \\ y' \\ z' \end{pmatrix}.$$

Put $x = z = 0$. Then for any $y \in K$, we have $ay = y' = ey$, $cy = z' = gy$ and $gy = 0$. Hence we have $a = e$ and $c = g = 0$.

Letting $x = y = 0$, for any $z \in K$ we have $bz = y' = 0$, $dz = z' = lz$ and $hz = 0$. Therefore $b = 0$, $d = l$ and $h = 0$. Combining these with $f + h = l$ and $a + b = c + d$, we have $a = d$ and $f = l = d$. Thus T_0 is a block diagonal such that

$$T_0 = a \oplus a \oplus a \oplus a \oplus a \oplus a \oplus a \oplus a.$$

Since T_0 is idempotent, a is also idempotent.

Finally consider that $H_4 = 0 \oplus 0 \oplus 0 \oplus K \oplus \{(y, Sy) \in K^2 \mid y \in K\}$ is invariant under T_0 . Then for any $x, y \in K$, there exist $x', y' \in K$ such that

$$T_0(0, 0, 0, x, y, Sy) = (0, 0, 0, ax, ay, aSy) = (0, 0, 0, x', y', Sy').$$

Hence $aSy = Sy' = Say$, so that $aS = Sa$. Since S is a unilateral shift and a is idempotent, we have $a = 0$ or $a = I$. This implies that $T_0 = 0$ or $T_0 = I$. Since for any $v \in V$ and any $x \in H_v$ $T_0x = T_vx$, we have $T_v = 0$ or $T_v = I$ simultaneously. Thus $T = (T_v)_{v \in V} = 0$ or $T = I$, that is, $\text{Idem}(H, f) = \{0, I\}$. Therefore (H, f) is indecomposable. \square

Remark. In many cases of our construction of indecomposable, infinite-dimensional representations, we can replace a unilateral shift S by any strongly irreducible operator.

We shall show that the existence of indecomposable, infinite-dimensional representations does not depend on the choice of the orientation of quivers. Suppose that two finite, connected quivers Γ and Γ' have the same underlying undirected graph and one of them, say Γ , has an infinite-dimensional, indecomposable, Hilbert representation. We need to prove that another quiver Γ' also has an infinite-dimensional,

indecomposable, Hilbert representation. Reflection functors are useful to show it. But we need to check the co-closedness at a source. We introduce a certain nice class of Hilbert representations such that co-closedness is easily checked and preserved under reflection functors at any source.

Definition Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n . We count the arrows from the left as $\alpha_k : s(\alpha_k) \rightarrow r(\alpha_k)$, ($k = 1, \dots, n-1$). Let (H, f) be a Hilbert representation of Γ . We denote f_{α_k} by f_k for short. For example,

$$\circ_{H_1} \xleftarrow{f_1} \circ_{H_2} \xrightarrow{f_2} \circ_{H_3} \xleftarrow{f_3} \circ_{H_4} \xrightarrow{f_4} \circ_{H_5} \xrightarrow{f_5} \circ_{H_6}$$

We say that (H, f) is *positive-unitary diagonal* if there exist $m \in \mathbb{N}$ and orthogonal decompositions (admitting zero components) of Hilbert spaces

$$H_k = \oplus_{i=1}^m H_{k,i} \quad (k = 1, \dots, n)$$

and decompositions of operators

$$f_k = \oplus_{i=1}^m f_{k,i} : \oplus_{i=1}^m H_{s(\alpha_k),i} \rightarrow \oplus_{i=1}^m H_{r(\alpha_k),i} \quad (k = 1, \dots, n),$$

such that each $f_{k,i} : H_{s(\alpha_k),i} \rightarrow H_{r(\alpha_k),i}$ is written as $f_{k,i} = 0$ or $f_{k,i} = \lambda_{k,i} u_{k,i}$ for some positive scalar $\lambda_{k,i}$ and onto unitary $u_{k,i} \in B(H_{s(\alpha_k),i}, H_{r(\alpha_k),i})$.

It is easy to see that if (H, f) is positive-unitary diagonal, then $\Phi^*(H, f)$ is also positive-unitary diagonal.

Example. (Inclusions of subspaces) Consider the following quiver Γ :

$$\circ_1 \xrightarrow{\alpha_1} \circ_2 \xrightarrow{\alpha_2} \circ_3$$

Let H_3 be a Hilbert space and $H_1 \subset H_2 \subset H_3$ inclusions of subspaces. Define a Hilbert representation (H, f) of Γ by $H = (H_i)_{i=1,2,3}$ and canonical inclusion maps $f_i = f_{\alpha_i} : H_i \rightarrow H_{i+1}$ for $i = 1, 2$. Then (H, f) is positive-unitary diagonal. In fact, define

$$K_1 = H_1, \quad K_2 = H_2 \cap H_1^\perp, \quad K_3 = H_3 \cap H_2^\perp.$$

Consider orthogonal decompositions $H_k = \oplus_{i=1}^3 H_{k,i}$ ($k = 1, 2, 3$) by

$$H_1 = K_1 \oplus 0 \oplus 0, \quad H_2 = K_1 \oplus K_2 \oplus 0 \quad \text{and} \quad H_3 = K_1 \oplus K_2 \oplus K_3.$$

Then $f_1 = I \oplus 0 \oplus 0$ and $f_2 = I \oplus I \oplus 0$. Hence (H, f) is positive-unitary diagonal. It is trivial that the example can be extended to the case of inclusion of n subspaces.

Lemma 6.4. *Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n and (H, f) be a Hilbert representation of Γ . Assume that (H, f) is positive-unitary diagonal. Then (H, f) is closed at any sink of Γ and co-closed at any source of Γ .*

Proof. Let v be a sink of Γ . Then $\sum_{\alpha \in E^v} \text{Im } f_\alpha$ is a finite sum of some of orthogonal subspaces $\{H_{v,i} \mid i\}$ of H_v which correspond to the images of positive times unitaries in the direct component of f_α . Hence it is a closed subspace of H_v . Therefore (H, f) is closed at v . Similarly (H, f) co-closed at any source of Γ . \square

Proposition 6.5. *Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n and (H, f) be a Hilbert representation of Γ . Let v be a source of Γ . Assume that (H, f) is positive-unitary diagonal. Then $\Phi_v^-(H, f)$ is also positive-unitary diagonal.*

Proof. If $(H, f) \cong (H', f') \oplus (H'', f'')$, then $\Phi_v^-(H, f) \cong \Phi_v^-(H', f') \oplus \Phi_v^-(H'', f'')$. Therefore $H_k^- = \bigoplus_{i=1}^m H_{k,i}^-$. Hence it is enough to consider orthogonal components. We may and do examine locally the following cases:

(Case 1): A Hilbert representation (H, f) is given by

$$\circ_{H_1} \xleftarrow{T_1} \circ_{H_0} \xrightarrow{T_2} \circ_{H_2}$$

with $T_1 = \lambda_1 U_1$ and $T_2 = \lambda_2 U_2$ for some positive scalars λ_1, λ_2 and onto unitaries U_1, U_2 . Put $(H^-, f^-) = \Phi_0^-(H, f)$:

$$\circ_{H_1} \xrightarrow{T_1^-} \circ_{H_0^-} \xleftarrow{T_2^-} \circ_{H_2}$$

Then $(a, b) \in H_1 \oplus H_2$ is in $H_0^- = (\text{Im } \hat{h}_0)^\perp$ if and only if $((a, b) \mid (T_1 z, T_2 z)) = 0$ for any $z \in H_0$, so that $T_1^* a + T_2^* b = 0$. Hence

$$\begin{aligned} H_0^- &= \{(a, -\lambda_1 \lambda_2^{-1} U_2 U_1^* a) \in H_1 \oplus H_2 \mid a \in H_1\} \\ &= \{(-\lambda_1^{-1} \lambda_2 U_1 U_2^* b, b) \in H_1 \oplus H_2 \mid b \in H_2\}. \end{aligned}$$

Solving

$$(x, 0) = (a, -\lambda_1 \lambda_2^{-1} U_2 U_1^* a) + (\lambda_1 U_1 z, \lambda_2 U_2 z) \in H_0^- \oplus \text{Im } \hat{h}_0,$$

we have

$$T_1^- x = \left(\frac{\lambda_2^2}{\lambda_1^2 + \lambda_2^2} x, -\frac{\lambda_1 \lambda_2}{\lambda_1^2 + \lambda_2^2} U_2 U_1^* x \right) \text{ for } x \in H_1.$$

Similarly we have

$$T_2^- y = \left(-\frac{\lambda_1 \lambda_2}{\lambda_1^2 + \lambda_2^2} U_1 U_2^* y, \frac{\lambda_2^2}{\lambda_1^2 + \lambda_2^2} y \right) \text{ for } y \in H_2.$$

Let $\lambda_1^- := \sqrt{(\frac{\lambda_2^2}{\lambda_1^2 + \lambda_2^2})^2 + (\frac{\lambda_1 \lambda_2}{\lambda_1^2 + \lambda_2^2})^2} > 0$ and $U_1^- := (\lambda_1^-)^{-1} T_1^-$. Then U_1^- is an onto unitary and $T_1^- = \lambda_1^- U_1^-$. Similarly T_2^- is a positive scalar times unitary.

(Case 2): A Hilbert representation (H, f) is given by

$$\circ_{H_1} \xleftarrow{T_1} \circ_{H_0} \xrightarrow{T_2} \circ_{H_2}$$

with $T_1 = 0$ and $T_2 = 0$

Then it is easy to see that $H_0^- = H_1 \oplus H_2$, T_1^- and T_2^- are canonical inclusions: $T_1^- x = (x, 0) \in H_1 \oplus H_2$ for $x \in H_1$ and $T_2^- y = (0, y) \in H_1 \oplus H_2$ for $y \in H_2$. We may write that $T_1^- = I \oplus 0 : H_1 \oplus 0 \rightarrow H_1 \oplus H_2$ and $T_2^- = 0 \oplus I : 0 \oplus H_2 \rightarrow H_1 \oplus H_2$. Hence (H^-, f^-) is positive-unitary diagonal.

(Case 3): A Hilbert representation (H, f) is given by

$$\circ_{H_1} \xleftarrow{T_1} \circ_{H_0} \xrightarrow{T_2} \circ_{H_2}$$

with $T_1 = \lambda_1 U_1$ and $T_2 = 0$ for some positive scalar λ_1 and onto unitary U_1 .

Then we see that $H_0^- = 0 \oplus H_2$, $T_1^- = 0$ and $T_2^- y = (0, y) \in 0 \oplus H_2$ for $y \in H_2$. Hence (H^-, f^-) is positive-unitary diagonal.

(Case 4): A Hilbert representation (H, f) is given by

$$\circ_{H_0} \xrightarrow{T_1} \circ_{H_1}$$

with $T_1 = \lambda_1 U_1$ for some positive scalar λ_1 and onto unitary U_1 . Put $(H^-, f^-) = \Phi_0^-(H, f)$:

$$\circ_{H_0^-} \xleftarrow{T_1^-} \circ_{H_1}$$

Then we see that $H_0^- = 0$ and $T_1^- = 0$.

(Case 5): A Hilbert representation (H, f) is given by

$$\circ_{H_0} \xrightarrow{T_1} \circ_{H_1}$$

with $T_1 = 0$.

Then we have that $H_0^- = H_1$ and $T_1^- = I : H_1 \rightarrow H_1 = H_0^-$. \square

We shall show that we can change the orientation of Dynkin diagram A_n using only the iteration of σ_v^- at sources v except the right end.

Lemma 6.6. *Let Γ_0 and Γ be quivers whose underlying undirected graphs are the same Dynkin diagram A_n for $n \geq 2$. We assume that Γ_0 is the following:*

$$\circ_1 \longrightarrow \circ_2 \longrightarrow \circ_3 \cdots \circ_{n-1} \longrightarrow \circ_n$$

Then there exists a sequence v_1, \dots, v_m of vertices in Γ_0 such that

- (1) *for each $k = 1, \dots, m$, v_k is a source in $\sigma_{v_{k-1}}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0)$,*
- (2) *$\sigma_{v_m}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0) = \Gamma$,*
- (3) *for each $k = 1, \dots, m$, $v_k \neq n$.*

Proof. The proof is by induction on the number n of vertices. Let $n = 2$. Since $\sigma_1^-(\circ_1 \longrightarrow \circ_2) = \circ_1 \longleftarrow \circ_2$, the statement holds. Assume that the statement holds for $n - 1$. If Γ has an arrow $\circ_{n-1} \longrightarrow \circ_n$, then we can directly apply the assumption of the induction. If Γ has an arrow $\circ_{n-2} \longrightarrow \circ_{n-1} \longleftarrow \circ_n$, replace only this part by $\circ_{n-2} \longleftarrow \circ_{n-1} \longrightarrow \circ_n$ to get Γ' . Then $n - 1$ is a source of Γ' , and $\sigma_{n-1}^-(\Gamma') = \Gamma$. Applying the induction assumption for Γ' , we can construct the desired iteration. Consider the case that Γ has an arrow $\circ_{n-2} \longleftarrow \circ_{n-1} \longleftarrow \circ_n$. If

there exist a vertex u such that $\circ_{u-1} \longrightarrow \circ_u$ and $\circ_k \longleftarrow \circ_{k+1}$ for $k = u, \dots, n-1$, then define a new quiver Γ'' by putting $\circ_{u-1} \longleftarrow \circ_u$, $\circ_{n-1} \longrightarrow \circ_n$ and other arrows unchanged with Γ . By the induction assumption, there exists a sequence v_1, \dots, v_m of vertices in Γ_0 such that $\sigma_{v_m}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0) = \Gamma''$ and, for each $k = 1, \dots, m$, $v_k \neq n$ and $v_k \neq n-1$. Then

$$\sigma_u^- \sigma_{u+1}^- \dots \sigma_{n-2}^- \sigma_{n-1}^- \sigma_{v_m}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0) = \Gamma.$$

If all the arrows between 1 and n are of the form $\circ_k \longleftarrow \circ_{k+1}$ for $k = 1, \dots, n-1$, then $\sigma_{n-1}^- \dots \sigma_2^- \sigma_1^- (\Gamma_0) = \Gamma$. \square

Lemma 6.7. *Let $\Gamma = (V, E, s, r)$ and $\Gamma' = (V', E', s', r')$ be finite, connected quivers and Γ' contains Γ as a subgraph, that is, $V \subset V'$, $E \subset E'$, $s = s'|_E$ and $r = r'|_E$. If there exists an infinite-dimensional, indecomposable, Hilbert representation of Γ , then there exists an infinite-dimensional, indecomposable, Hilbert representation of Γ' .*

Proof. Let (H, f) be an infinite-dimensional, indecomposable, Hilbert representation of Γ . Define $H'_v = H_v$ for $v \in V$ and $H'_v = 0$ for $v \in V' \setminus V$. We put $f'_\alpha = f_\alpha$ for $\alpha \in E$ and $f'_\alpha = 0$ for $\alpha \in E' \setminus E$. Then it is clear that (H', f') is an infinite-dimensional, indecomposable, Hilbert representation of Γ' . \square

We are ready to prove our main theorem.

Theorem 6.8. *Let Γ be a finite, connected quiver. If the underlying undirected graph $|\Gamma|$ contains one of the extended Dynkin diagrams \tilde{A}_n ($n \geq 0$), \tilde{D}_n ($n \geq 4$), \tilde{E}_6 , \tilde{E}_7 and \tilde{E}_8 , then there exists an infinite-dimensional, indecomposable, Hilbert representation of Γ .*

Proof. By Lemma 6.7, we may assume that the underlying undirected graph $|\Gamma|$ is exactly one of the extended Dynkin diagrams \tilde{A}_n ($n \geq 0$), \tilde{D}_n ($n \geq 4$), \tilde{E}_6 , \tilde{E}_7 and \tilde{E}_8 .

The case of extended Dynkin diagrams \tilde{A}_n ($n \geq 0$) was already verified in Example 1 and 2 in section 3.

Next suppose that $|\Gamma|$ is \tilde{E}_6 . Let Γ_0 be the quiver of Example 4 in section 3 and we denote here by $(H^{(0)}, f^{(0)})$ the Hilbert representation constructed there. Then $|\Gamma_0| = |\Gamma| = \tilde{E}_6$, but their orientations are different in general. Three "wings" of $|\Gamma_0|$ $2-1-0$, $2'-1'-0$, $2''-1''-0$ are of A_3 . Applying Lemma 6.6 for these wings locally, we can find a sequence v_1, \dots, v_m of vertices in Γ_0 such that

- (1) for each $k = 1, \dots, m$, v_k is a source in $\sigma_{v_{k-1}}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0)$,
- (2) $\sigma_{v_m}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0) = \Gamma$,
- (3) for each $k = 1, \dots, m$, $v_k \neq 0$.

We note that co-closedness of Hilbert representations at a source can be checked locally around the source. Since the restriction of the representation $(H^{(0)}, f^{(0)})$ to each "wing" is positive-unitary diagonal and the

iteration of reflection functors does not move the vertex 0, we can apply Lemma 6.4 and Proposition 6.5 locally that $\Phi_{v_{k-1}}^- \dots \Phi_{v_2}^- \Phi_{v_1}^- (H^{(0)}, f^{(0)})$ is co-closed at v_k for $k = 1, \dots, m$. Therefore Theorem 5.13 implies that $(H, f) := \Phi_{v_m}^- \dots \Phi_{v_2}^- \Phi_{v_1}^- (H^{(0)}, f^{(0)})$ is the desired indecomposable, Hilbert representation of Γ . Since the particular Hilbert space $H_0^{(0)}$ associated with the vertex 0 is infinite dimensional and remains unchanged under the iteration of the reflection functors above, (H, f) is infinite dimensional.

The case that the $|\Gamma|$ is \tilde{E}_7 or \tilde{E}_8 is shown similarly if we apply iteration of reflection functors on the representations in Lemma 6.2 or Lemma 6.3.

Finally consider the case that the $|\Gamma|$ is \tilde{D}_n . Let Γ_0 be the quiver of Lemma 6.1 and $(H^{(0)}, f^{(0)})$ the Hilbert representation constructed there. Then $|\Gamma_0| = |\Gamma| = \tilde{D}_n$, but their orientations are different in general. Let Γ_1 be a quiver such that $|\Gamma_1| = \tilde{D}_n$ and the orientation is as same as Γ on the path between 5 and $n+1$ and as same as Γ_0 on the rest four "wings". Define a Hilbert representation $(H^{(1)}, f^{(1)})$ of Γ_1 similarly as $(H^{(0)}, f^{(0)})$. For any arrow β in the path between 5 and $n+1$, $f_\beta^{(1)} = I$. Hence the same proof as for $(H^{(0)}, f^{(0)})$ shows that $(H^{(1)}, f^{(1)})$ is indecomposable. By a certain iteration of reflection functors at a source 1, 2, 3 or 4 on $(H^{(1)}, f^{(1)})$ yields an infinite-dimensional, indecomposable, Hilbert representation of Γ . Here the co-closedness at a source 1, 2, 3 or 4 on $(H^{(1)}, f^{(1)})$ is easily checked, because the map is the canonical inclusion. Thus we can apply Theorem 5.13 in this case too. \square

Corollary 6.9. *Let Γ be a finite, connected quiver. If there exists no infinite-dimensional, indecomposable, Hilbert representation of Γ , then the underlying undirected graph $|\Gamma|$ is one of the Dynkin diagrams A_n ($n \geq 1$), D_n ($n \geq 4$), E_6 , E_7 and E_8 .*

Proof. It directly follows from a well known fact that if the underlying undirected graph $|\Gamma|$ contains no extended Dynkin diagrams, then $|\Gamma|$ is one of the Dynkin diagrams. \square

Remark. We have not yet proved the converse. In fact if the converse were true, then a long standing problem on transitive lattices of subspaces of Hilbert spaces would be settled. Recall that Halmos initiated the study of transitive lattices and gave an example of transitive lattice consisting of seven subspaces in [Ha]. Harrison-Radjavi-Rosenthal [HRR] constructed a transitive lattice consisting of six subspaces using the graph of an unbounded closed operator. Hadwin-Longstaff-Rosenthal found a transitive lattice of five non-closed linear subspaces in [HLR]. Any finite transitive lattice which consists of n subspaces of a Hilbert space H gives an indecomposable system of $n - 2$ subspaces by withdrawing 0 and H . It is still unknown whether or not there exists

a transitive lattice consisting of five subspaces. Therefore it is also an interesting problem to know whether there exists an indecomposable system of three subspaces in an infinite-dimensional Hilbert space. The problem can be rephrased as whether there exists an indecomposable representation of a certain quiver whose underlying undirected graph is D_4 in an infinite-dimensional Hilbert space.

We have a partial evidence for a certain quiver whose underlying undirected graph is A_n . We prepare an elementary lemma. Let H be a Hilbert space. For $a, b \in H$ we denote by $\theta_{a,b}$ a rank one operator on H such that $\theta_{a,b}(x) = (x|b)a$ for $x \in H$. Then $\theta_{a,b}^2 = \theta_{a,b}$ if and only if $(a|b) = 1$ or $a = 0$ or $b = 0$. Moreover if $\dim H \geq 2$ and $(a|b) = 1$, then $\theta_{a,b}$ is an idempotent such that $\theta_{a,b} \neq 0$ and $\theta_{a,b} \neq I$.

Lemma 6.10. *Let H_1 and H_2 be Hilbert spaces and $T : H_1 \rightarrow H_2$ a bounded operator. Take $a, b \in H_1$ and $c, d \in H_2$. Suppose that there exists a scalar λ such that $Ta = \lambda c$ and $T^*d = \bar{\lambda}b$. Then $T\theta_{a,b} = \theta_{c,d}T$*

Proof.

$$T\theta_{a,b} = \theta_{Ta,b} = \theta_{\lambda c,b} = \theta_{c,\bar{\lambda}b} = \theta_{c,T^*d} = \theta_{c,d}T.$$

□

Proposition 6.11. *Let Γ be the following quiver whose underlying undirected graph is A_n for $n \geq 1$:*

$$\circ_1 \xrightarrow{\alpha_1} \circ_2 \xrightarrow{\alpha_2} \circ_3 \cdots \circ_{n-1} \xrightarrow{\alpha_{n-1}} \circ_n$$

Then there exists no infinite-dimensional, indecomposable, Hilbert representation of Γ .

Proof. The case $n = 1$ is clear by a nontrivial decomposition $H_1 = L_1 \oplus K_1$. We may assume that $n \geq 2$. Suppose that there were an infinite-dimensional, indecomposable, Hilbert representation (H, f) of Γ . Put $T_k = f_{\alpha_k} : H_k \rightarrow H_{k+1}$ for $k = 1, \dots, n-1$.

(case 1) Suppose that $T_{n-1}T_{n-2}\dots T_1 \neq 0$. Then there exists $a_1 \in H_1$ such that $T_{n-1}T_{n-2}\dots T_1 a_1 \neq 0$. Consider non-zero vectors $a_k = T_{k-1}T_{k-2}\dots T_1 a_1 \in H_k$ for $k = 1, \dots, n$. Put $b_n = \|a_n\|^{-2} a_n \in H_n$. Define $b_i = T_i^* T_{i+1}^* \dots T_{n-1}^* b_n \in H_i$ for $i = 1, 2, \dots, n-1$. Then

$$(a_i|b_i) = (a_i|T_i^* T_{i+1}^* \dots T_{n-1}^* b_n) = (T_{n-1}T_{n-2}\dots T_i a_i|b_n) = (a_n|b_n) = 1.$$

Since $T_k a_k = a_{k+1}$ and $T_k^* b_{k+1} = b_k$, the above Lemma 6.10 implies that $T_k \theta_{a_k, b_k} = \theta_{a_{k+1}, b_{k+1}} T_k$ for $k = 1, \dots, n-1$. Define the non-zero idempotents $P_k = \theta_{a_k, b_k}$. Since (H, f) is infinite dimensional, there exists some vertex m such that H_m is infinite dimensional. Then $P_m \neq I$. Define $P = (P_k)_k$, then $P \in \text{Idem}(H, f)$ and $P \neq O$ and $P \neq I$. This contradicts the assumption that (H, f) is indecomposable.

(case 2) Suppose that there exists r such that $T_{r-1}T_{r-2}\dots T_1 \neq 0$ and $T_r T_{r-1}\dots T_1 = 0$ for some $r = 1, \dots, n-1$ and $\dim H_m \geq 2$ for some $m = 1, \dots, r$. Then there exists $a_1 \in H_1$ such that $T_{r-1}T_{r-2}\dots T_1 a_1 \neq 0$

Consider non-zero vectors $a_k = T_{k-1}T_{k-2}\dots T_1a_1 \in H_k$ for $k = 1, \dots, r$. Put $b_r = \|a_r\|^{-2}a_r \in H_r$. Define $b_i = T_i^*T_{i+1}^*\dots T_{r-1}^*b_r \in H_i$ for $i = 1, 2, \dots, r-1$. Then we have $T_k\theta_{a_k,b_k} = \theta_{a_{k+1},b_{k+1}}T_k$ for $k = 1, \dots, r-1$ as case 1. Define non-zero idempotents $P_k = \theta_{a_k,b_k}$ for $k = 1, \dots, r$. Put $P_k = 0$ for $k = r+1, \dots, n$. Then $T_r\theta_{a_r,b_r} = \theta_{T_ra_r,b_r} = \theta_{0,b_r} = 0$ and $T_kP_k = P_{k+1}T_k = 0$ for $k = r, \dots, n-1$. Since $\dim H_m \geq 2$, the non-zero idempotent $P_m \neq I$. Define $P = (P_k)_k$, then $P \in \text{Idem}(H, f)$ and $P \neq O$ and $P \neq I$. This is a contradiction.

(case 3) Suppose that there exists r such that $T_{r-1}T_{r-2}\dots T_1 \neq 0$ and $T_rT_{r-1}\dots T_1 = 0$ for some $r = 1, \dots, n$ and $\dim H_k = 1$ for $k = 1, \dots, r$. Therefore $T_r = 0$. We may put $P_k = 0$ for $k = 1, \dots, r$. Then for any $a, b \in H_{r+1}$ and $P_{r+1} = \theta_{a,b}$, we have $T_kP_k = P_{k+1}T_k = 0$ for $k = 1, \dots, r$. Hence we may choose freely P_k for $k = r+1, \dots, n$. Starting from H_{r+1} , we can repeat the argument from the beginning. After finite steps, we can reduce to the situation of case 1 or case 2. And finally we obtain a contradiction. \square

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